

Studies on 1/4 BPS and 1/8 BPS geometries

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Abstract

We analyze more explicitly four sectors of 1/4 BPS geometries and 1/8 BPS geometries, corresponding to BPS states in $\mathcal{N}=4$ SYM, constructed previously. These include the states with several $SO(6)$ angular momenta as well as those with $SO(4)$ AdS spins. We also discuss their relations to the dual gauge theory.

1 Introduction

In this paper, we study certain sectors of 1/4 BPS and 1/8 BPS geometries in type IIB string theory.

The 1/2 BPS geometries corresponding to a class of 1/2 BPS states in $\mathcal{N}=4$ SYM were studied in [1] - [4]. Many geometric and topological aspects of the geometries were discussed in e.g. [1] - [26]. Geometries corresponding to Wilson loops, surface operators, and defect operators were constructed and discussed in e.g. [39] - [47], [11] - [13].

On the other hand, 1/4 BPS and 1/8 BPS geometries were studied by e.g. [27] - [35]. For the 1/4 BPS and 1/8 BPS geometries, we will in this paper, continue to analyze more detailed explicit geometries, and their relation to the dual gauge theory. We will also study separately the four different sectors of geometries.

The organization of this paper is as follows. In section 2, we analyze geometries with angular momenta J_1, J_2 in S^5 directions. In section 3, we analyze geometries with J_1, J_2, J_3 in S^5 directions. In section 4, we analyze the geometries with spin S_1 in AdS_5 and J in S^5 directions respectively. In section 5, we analyze geometries with spins S_1, S_2 in AdS_5 and J in S^5 directions respectively. Finally, we briefly conclude in section 6.

2 1/4 BPS geometries with J_1, J_2

2.1 General ansatz

In this section we study the 1/4 BPS states with $U(1)_t \times SO(4) \times SO(2)$ symmetry. The geometries corresponding to such BPS states have been studied in e.g. [27] - [33], which is a $R_t \times S^3 \times S^1$ fibration over 4d Kähler base. In addition, there is a direction y , as the product of two radii of S^3 and S^1 . As argue in [30], the S^3 and S^1 shrinks smoothly on two types of droplet regions in the 4d base as the direction y goes to zero, i.e. $y = 0$.

We first discuss the ansatz for the 1/4 BPS configurations. These backgrounds have an additional S^1 isometry compared with the 1/8 BPS backgrounds, and have a ten-dimensional solution of the form (in the conventions of [28], [30], [33]),

$$\begin{aligned}
ds_{10}^2 &= -h^{-2}(dt + \omega)^2 + h^2((Z + \frac{1}{2})^{-1}2\partial_i\partial_{\bar{j}}K dz^i d\bar{z}^{\bar{j}} + dy^2) + y(e^G d\Omega_3^2 + e^{-G}(d\psi + \mathcal{A})^2), \\
F_5 &= \{-d[y^2 e^{2G}(dt + \omega)] - y^2(d\omega + \eta\mathcal{F}) + 2i\partial\bar{\partial}K\} \wedge d\Omega^3 + \text{dual}, \\
h^{-2} &= 2y \cosh G, \\
Z &= \frac{1}{2}\tanh G = -\frac{1}{2}y\partial_y(\frac{1}{y}\partial_y K), \\
d\omega &= \frac{i}{2}d[\frac{1}{y}\partial_y(\bar{\partial} - \partial)K] = \frac{i}{y}(\partial_i\partial_{\bar{j}}\partial_y K dz_i d\bar{z}_{\bar{j}} + \partial_i Z d\bar{z}_i dy - \partial_i Z dz_i dy), \\
2\eta\mathcal{F} &= -i\partial\bar{\partial}D.
\end{aligned} \tag{2.1}$$

The $K = K(z_i, \bar{z}_i; y)$, where $i = 1, 2$, is the Kähler potential for the 4d base, which also varies with the y direction. D is an auxiliary function, and can be set to a constant, if the fibration of the S^1 is a direct product, i.e. $\mathcal{A} = 0$. The volume of the 4d base is constrained by a Monge-Ampere equation (as well as an auxiliary condition for auxiliary function D),

$$\log \det h_{i\bar{j}} = \log(Z + \frac{1}{2}) + n\eta \log y + \frac{1}{y}(2 - n\eta)\partial_y K + D(z_i, \bar{z}_j), \quad (2.2)$$

$$(1 + *_4)\partial\bar{\partial}D = \frac{4}{y^2}(1 - n\eta)\partial\bar{\partial}K. \quad (2.3)$$

In other words,

$$\det \partial_i \partial_{\bar{j}} K = (Z + \frac{1}{2}) y^{n\eta} e^{\frac{1}{y}(2 - n\eta)\partial_y K} e^D. \quad (2.4)$$

We can consider two types of 1/4 BPS states with the above ansatz. One class is the states with two R-charges J_1, J_2 , which will be discussed in this section. Another class is the states with R-charge J and an AdS spin or an $SO(4)$ spin S_1 , and will be discussed in section 4. The first case corresponds to that the S^3 in the ansatz (2.1) is in the AdS directions, while the second case corresponds to that the S^3 in the ansatz (2.1) is in the S^5 directions.

For the first 1/4 BPS sector, the dual operator is of the schematic form

$$O \sim \prod_{i=1}^m \text{tr}(Z^{n_{1i}} Y^{n_{2i}}) \quad (2.5)$$

where Z, Y are two complex scalars of $\mathcal{N}=4$ SYM. The BPS bound is satisfied as

$$\Delta - J_1 - J_2 = 0. \quad (2.6)$$

2.2 Small y analysis

For simplicity of the discussion, we first consider the case when the S^1 is a direct product factor, so we can set $\mathcal{A} = 0, \mathcal{F} = d\mathcal{A} = 0$, as well as $n\eta = 1$. We can also set D as a constant in (2.4). The equation is

$$\det \partial_i \partial_{\bar{j}} K = (1 - 4y^2 \partial_{y^2}^2 K) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} K} \quad (2.7)$$

where we used $e^D = \frac{1}{4\sqrt{e}}$, and take derivatives with respect to the y^2 .

In this subsection we analyze the equations from (2.7) as well as equations of regularity conditions at small y . We denote I_- as the region $S^3 \rightarrow 0, Z = -\frac{1}{2}$; and I_+ as the region $S^1 \rightarrow 0, Z = \frac{1}{2}$. The small y behavior of the Kähler potential near two different types of droplet regions are very different, and we will analyze the small y equations in each droplet regions.

In the region that S^1 shrinks, we have

$$Z = \frac{1}{2}, \quad \text{i.e.} \quad -\frac{1}{2} y \partial_y \left(\frac{1}{y} \partial_y K \right) \Big|_{y=0} = \frac{1}{2} \quad (2.8)$$

where the 1/2 BPS function Z is related to K via $Z = -\frac{1}{2}y\partial_y(\frac{1}{y}\partial_y K)$, and we have $h^2(Z + \frac{1}{2})^{-1} = O(1)$ in the $Z = \frac{1}{2}$ region. The leading terms in K that have z_i, \bar{z}_i dependence will go as $K_0(z_i, \bar{z}_i) + y^2 K_1(z_i, \bar{z}_i)$. In this region one can expand the K as

$$K = -\frac{1}{4}y^2 \log(y^2) + K_0(z_i, \bar{z}_i) + y^2 K_1(z_i, \bar{z}_i) + (y^2)^2 K_2(z_i, \bar{z}_i) \quad (2.9)$$

i.e. up to order $(y^2)^3$. We also have the expansion of $Z = \frac{1}{2} - 4y^2 K_2$.

In the region that S^3 shrinks, we have $Z = -\frac{1}{2}$, and $h^{-2}(Z + \frac{1}{2}) = o(y^2)$, so from the metric we require $\partial_i \partial_{\bar{j}} K|_{y=0} = 0$. Thereby in this region,

$$Z = -\frac{1}{2}, \quad \text{i.e.} \quad -\frac{1}{2}y\partial_y(\frac{1}{y}\partial_y K)|_{y=0} = -\frac{1}{2}, \quad \text{and} \quad \partial_i \partial_{\bar{j}} K|_{y=0} = 0. \quad (2.10)$$

One can expand the K in this region as

$$K = \frac{1}{4}y^2 \log(y^2) + K_0(z_i, \bar{z}_i) + y^2 K_1(z_i, \bar{z}_i) + (y^2)^2 K_2(z_i, \bar{z}_i), \quad (2.11)$$

up to order $(y^2)^3$, with additional equation $\partial_i \partial_{\bar{j}} K_0 = 0$, due to $\partial_i \partial_{\bar{j}} K|_{y=0} = 0$. A necessary but weaker condition is $\det \partial_i \partial_{\bar{j}} K_0 = 0$ [30]. We also have the expansion of $Z = -\frac{1}{2} - 4y^2 K_2$.

We then analyze the equations for K_0, K_1, K_2 in those two regions at $y=0$.

While in I_- i.e. $S^3 \rightarrow 0$, $Z = -\frac{1}{2}$, from (2.7),

$$\det \partial_i \partial_{\bar{j}} K_0 = 0, \quad (2.12)$$

$$\partial_i \partial_{\bar{j}} K_1 \partial_k \partial_{\bar{l}} K_0 (\delta^{i\bar{j}} \delta^{k\bar{l}} - \delta^{i\bar{l}} \delta^{k\bar{j}}) = 0. \quad (2.13)$$

In addition, from the regularity condition

$$\partial_i \partial_{\bar{j}} K_0 = 0. \quad (2.14)$$

These equations can be simultaneously solved by $\partial_i \partial_{\bar{j}} K_0 = 0$.

Assuming $\partial_i \partial_{\bar{j}} K_0 = 0$, which is a stronger condition than $\det \partial_i \partial_{\bar{j}} K_0 = 0$, then from (2.7)

$$\det \partial_i \partial_{\bar{j}} K_1 + K_2 e^{2K_1} = 0. \quad (2.15)$$

This suggests that K_2 can be determined once knowing the expression of K_1 , i.e.

$$K_2 = -e^{-2K_1} \det \partial_i \partial_{\bar{j}} K_1. \quad (2.16)$$

While in I_+ i.e. $S^1 \rightarrow 0$, $Z = \frac{1}{2}$, we get from (2.7),

$$\det \partial_i \partial_{\bar{j}} K_0 = \frac{1}{4e} e^{2K_1}, \quad (2.17)$$

$$\partial_i \partial_{\bar{j}} K_1 \partial_k \partial_{\bar{l}} K_0 (\delta^{i\bar{j}} \delta^{k\bar{l}} - \delta^{i\bar{l}} \delta^{k\bar{j}}) = 0. \quad (2.18)$$

The first equation can be written as

$$K_1 = \frac{1}{2} \log \det \partial_i \partial_{\bar{j}} K_0 + \log 2 + \frac{1}{2}. \quad (2.19)$$

To summarize a little, in the I_- region, we have equations (2.14),(2.13) for K_0, K_1 , and (2.16) serves as the solution for K_2 . While in the I_+ region, we have the coupled equations (2.17),(2.18) for K_0, K_1 which can be solved.

One can also argue that in the limit that I_- regions have zero measure in terms of the 4d volume, the equations (2.17),(2.18) are valid in all the regions in 4d, except the loci of I_- .

We now study how to solve these equations. We first study the I_+ region. We can expand in the large $r^2 = |z_1|^2 + |z_2|^2$ region as

$$K_0 = \frac{1}{2}(|z_1|^2 + |z_2|^2) + \tilde{K}_0, \quad K_1 = \frac{1}{2} + \tilde{K}_1. \quad (2.20)$$

From the leading order terms in the tilded variables we have

$$\partial_1 \partial_{\bar{1}} \tilde{K}_0 + \partial_2 \partial_{\bar{2}} \tilde{K}_0 = \tilde{K}_1, \quad (2.21)$$

$$\partial_1 \partial_{\bar{1}} \tilde{K}_1 + \partial_2 \partial_{\bar{2}} \tilde{K}_1 = 0. \quad (2.22)$$

The expression of the \tilde{K}_0 can always be obtained from \tilde{K}_1 using the Green's function technique, due to equation (2.21).

So we get in leading orders in large $|z_1|^2 + |z_2|^2$ region, using 4d Green's function method,

$$K_0 = \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \int_{\mathcal{D}} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2) u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2, \quad (2.23)$$

$$K_1 = \frac{1}{2} - \frac{1}{2} \int_{\mathcal{D}} \frac{1}{(|z_1 - z'_1|^2 + |z_2 - z'_2|^2)} u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2, \quad (2.24)$$

where $\int_{\mathcal{D}} u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 = 1$. For convenience we use a notation that $d^2 z'_1 = \frac{1}{2i} dz'_1 d\bar{z}'_1 = dx' dy'$, and (z'_i, \bar{z}'_i) denotes $(z'_1, \bar{z}'_1, z'_2, \bar{z}'_2)$. The logarithmic terms in K_0 in (2.23) is very reminiscent of the repulsions between eigenvalues in the droplet space.

Now we alternatively expand around a region near a particular radial position, e.g. near $|z_1|^2 + |z_2|^2 \simeq 1$, as

$$K_0 = \frac{1}{2} + \frac{1}{4}(|z_1|^2 + |z_2|^2 - 1)^2 + \tilde{K}_0, \quad K_1 = \frac{1}{2} \log(|z_1|^2 + |z_2|^2 - 1) + \frac{1}{2} + \tilde{K}_1. \quad (2.25)$$

After taking $|z_1|^2 + |z_2|^2 \rightarrow 1$ limit, we get the leading order equations

$$|z_2|^2 \partial_1 \partial_{\bar{1}} \tilde{K}_0 + |z_1|^2 \partial_2 \partial_{\bar{2}} \tilde{K}_0 - z_1 \bar{z}_2 \partial_1 \partial_{\bar{2}} \tilde{K}_0 - z_2 \bar{z}_1 \partial_2 \partial_{\bar{1}} \tilde{K}_0 = 0, \quad (2.26)$$

$$|z_2|^2 \partial_1 \partial_{\bar{1}} \tilde{K}_1 + |z_1|^2 \partial_2 \partial_{\bar{2}} \tilde{K}_1 - z_1 \bar{z}_2 \partial_1 \partial_{\bar{2}} \tilde{K}_1 - z_2 \bar{z}_1 \partial_2 \partial_{\bar{1}} \tilde{K}_1 = 0. \quad (2.27)$$

We see that $\tilde{K}_1 = -\frac{1}{2}(|z_1|^2 + |z_2|^2 - 1)$ is an exact solution to the equation, and $\tilde{K}_0 \simeq -\frac{1}{6}(|z_1|^2 + |z_2|^2 - 1)^3$ is a leading order solution.

The equation is not exactly superposable, but in the small $|z'_1|^2 + |z'_2|^2 \ll 1$ limit, it is approximately superposable, so in that limit,

$$K_1 \simeq \frac{1}{2} \log(|z_1|^2 + |z_2|^2 - 1) + 1 - \frac{1}{2} \int_{\mathcal{D}} (|z_1 - z'_1|^2 + |z_2 - z'_2|^2) u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 \quad (2.28)$$

where $\int_{\mathcal{D}} u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 = 1$. This means that there are extra small $Z = -\frac{1}{2}$ droplets located at (z'_i, \bar{z}'_i) , which are close to the origin and far from $|z_1|^2 + |z_2|^2 = 1$, in which case the superposition is possible.

2.3 Small variation

In this subsection we describe a change of variable that transforms (2.7) into linear equation, under the approximation that the changed variable is slowly varying.

We consider the change of variable

$$K(z_i, \bar{z}_i, y) = \frac{1}{2}y^2 - \frac{1}{4}y^2 \log y^2 + \frac{1}{2}(|z_1|^2 + |z_2|^2) + V(z_i, \bar{z}_i, y) \quad (2.29)$$

and $V(z_i, \bar{z}_i, y)$ is a new function.

We have

$$\partial_i \partial_{\bar{j}} K = \frac{1}{2} \delta_{i\bar{j}} + \partial_i \partial_{\bar{j}} V. \quad (2.30)$$

Then (2.7) becomes

$$\frac{1}{4} + \frac{1}{2} \partial_1 \partial_{\bar{1}} V + \frac{1}{2} \partial_2 \partial_{\bar{2}} V + (\partial_1 \partial_{\bar{1}} V \partial_2 \partial_{\bar{2}} V - \partial_1 \partial_{\bar{2}} V \partial_2 \partial_{\bar{1}} V) = \frac{1}{4} e^{2\partial_y^2 V} (1 - 2y^2 \partial_y^2 V). \quad (2.31)$$

Now if we assume slowly varying V

$$\partial_i V, \partial_{\bar{j}} V, \partial_y^2 V \ll 1, \quad (2.32)$$

we then get

$$4(\partial_1 \partial_{\bar{1}} V + \partial_2 \partial_{\bar{2}} V) + y^3 \partial_y \left(\frac{1}{y^3} \partial_y V \right) = 0. \quad (2.33)$$

One can see that $V = -\frac{1}{2} \log(|z_1|^2 + |z_2|^2 + y^2)$ are $V = \frac{y^4}{(|z_1|^2 + |z_2|^2 + y^2)^4}$ are solutions to this linear and superposable equation.

One way to treat the equation is change of variable

$$\Psi = \frac{V}{y^2}, \quad (2.34)$$

$$4(\partial_1 \partial_{\bar{1}} \Psi + \partial_2 \partial_{\bar{2}} \Psi) + \frac{1}{y} \partial_y (y \partial_y \Psi) = \frac{4}{y^2} \Psi. \quad (2.35)$$

The equation (2.35) is a Poisson equation in 6d.

The solutions to (2.33) can be written as

$$V = \int_{\mathcal{D}} \left(-\frac{1}{2} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2 + y^2) \right) u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 + \int_{\mathcal{D}} \frac{\alpha y^4 u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2}{(|z_1 - z'_1|^2 + |z_2 - z'_2|^2 + y^2)^4} \quad (2.36)$$

where $\int_{\mathcal{D}} u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 = 1$.

The solution is valid in the region of slowly varying V , which includes $y = 0, Z = +\frac{1}{2}$, as one can check that (2.29), (2.36) satisfies the boundary condition (2.8). It also includes large y region. The solution breaks down near the $y = 0, Z = -\frac{1}{2}$ droplets.

The Kahler potential can be written through (2.29). In particular the Kahler potential at $y = 0$ can be written as

$$K|_{y=0} = \frac{1}{2}(|z_1|^2 + |z_2|^2) + \int_{\mathcal{D}} \left(-\frac{1}{2} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2) \right) u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2, \quad Z = +\frac{1}{2} \quad (2.37)$$

up to an overall constant shift. The $Z = -\frac{1}{2}$ droplets are like conducting droplets in the 4d base.

2.4 Radially symmetric cases

In this subsection, we look at the special case that $K = K(r^2, y^2)$, which means that the Kähler potential only depends on the radial direction $r^2 = |z_1|^2 + |z_2|^2$ of the 4d base and y .

The equation from (2.7) is (where we have introduced $e^D = \frac{1}{4\sqrt{e}}$)

$$\partial_{r^2} K \partial_{r^2} (r^2 \partial_{r^2} K) = (1 - 4y^2 \partial_{y^2}^2 K) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} K} \quad (2.38)$$

where the derivatives are taken with respect to r^2 and y^2 .

We try to look for general solutions that S^3 shrinks at several intervals in the r direction, i.e.

$$I_- = (0, r_1) \cup (r_2, r_3) \cup \dots \cup (r_{2m}, r_{2m+1}) \quad (2.39)$$

with $Z = -\frac{1}{2}$; while S^1 shrinks at

$$I_+ = (r_1, r_2) \cup (r_3, r_4) \cup \dots \cup (r_{2m+1}, \infty) \quad (2.40)$$

with $Z = \frac{1}{2}$. Each region in I_- (except the first) is a ‘shell’ of $Z = -\frac{1}{2}$ droplet. Each region in I_+ (except the last) is a ‘shell’ of $Z = \frac{1}{2}$ droplet. The superposable solutions in small y in subsection 2.2 indicates that the above configurations likely exist.

The $AdS_5 \times S^5$ is an exact solution to this equation (2.38) as follows [30]

$$\begin{aligned} K_{AdS} = & \frac{1}{2} \left(\frac{1}{2}(r^2 + y^2 + 1) + \sqrt{\frac{1}{4}(r^2 + y^2 - 1)^2 + y^2} \right) \\ & - \frac{1}{2} \log \left(\frac{1}{2}(r^2 + y^2 + 1) + \sqrt{\frac{1}{4}(r^2 + y^2 - 1)^2 + y^2} \right) \\ & - \frac{1}{2} y^2 \log \left(\frac{1}{2}(-r^2 + y^2 + 1) + \sqrt{\frac{1}{4}(r^2 + y^2 - 1)^2 + y^2} \right) + \frac{1}{4} y^2 \log y^2 \quad (2.41) \end{aligned}$$

and the Z is

$$Z = \frac{1}{2} \frac{r^2 + y^2 - 1}{\sqrt{(r^2 + y^2 - 1)^2 + 4y^2}} \quad (2.42)$$

where the variables are written in the unit that the AdS radius $L = 1$. For AdS , $I_- = (0, 1)$ and $I_+ = (1, \infty)$.

To get more intuition, we first look at the small y expansion of the $AdS_5 \times S^5$ solution: When $S^3 \rightarrow 0$, $Z = -\frac{1}{2}$, i.e. $r^2 < 1$,

$$K = \frac{1}{4}y^2 \log y^2 + K_0 + y^2 K_1 + (y^2)^2 K_2 + o((y^2)^3), \quad (2.43)$$

$$K_0 = \frac{1}{2}, \quad K_1 = -\frac{1}{2} \log(1 - r^2), \quad K_2 = -\frac{1}{4(r^2 - 1)^2}. \quad (2.44)$$

On the other hand, when $S^1 \rightarrow 0$, $Z = \frac{1}{2}$, i.e. $r^2 > 1$,

$$K = -\frac{1}{4}y^2 \log y^2 + K_0 + y^2 K_1 + (y^2)^2 K_2 + o((y^2)^3), \quad (2.45)$$

$$K_0 = \frac{1}{2}(r^2 - \log r^2), \quad K_1 = \frac{1}{2}(1 + \log(r^2 - 1) - \log r^2), \quad K_2 = \frac{1}{4(r^2 - 1)^2}. \quad (2.46)$$

The small y equations for K with radial symmetry are:

When $r \in I_-$ i.e. $S^3 \rightarrow 0$, $Z = -\frac{1}{2}$,

$$\partial_{r^2} K_0 (\partial_{r^2} K_0 + r^2 \partial_{r^2}^2 K_0) = 0, \quad (2.47)$$

$$\partial_{r^2} K_0 (\partial_{r^2} K_1 + r^2 \partial_{r^2}^2 K_1) + \partial_{r^2} K_1 (\partial_{r^2} K_0 + r^2 \partial_{r^2}^2 K_0) = 0, \quad (2.48)$$

$$\partial_{r^2} K_0 (\partial_{r^2} K_2 + r^2 \partial_{r^2}^2 K_2) + \partial_{r^2} K_1 (\partial_{r^2} K_1 + r^2 \partial_{r^2}^2 K_1) + K_2 e^{2K_1} = 0. \quad (2.49)$$

When $r \in I_+$ i.e. $S^1 \rightarrow 0$, $Z = \frac{1}{2}$,

$$\partial_{r^2} K_0 (\partial_{r^2} K_0 + r^2 \partial_{r^2}^2 K_0) - \frac{1}{4e} e^{2K_1} = 0, \quad (2.50)$$

$$\partial_{r^2} K_0 (\partial_{r^2} K_1 + r^2 \partial_{r^2}^2 K_1) + \partial_{r^2} K_1 (\partial_{r^2} K_0 + r^2 \partial_{r^2}^2 K_0) = 0. \quad (2.51)$$

We see that the AdS expression is one solution to the equations.

We first look at the coupled equations for K_0, K_1 in region $r \in I_+$:

$$(K'_0)^2 = -\frac{1}{8e} (e^{2K_1})' \frac{1}{K'_1 + r^2 K''_1}, \quad (2.52)$$

$$r^2 K'_0 K''_0 = \frac{1}{4e} (e^{2K_1} + \frac{1}{2} (e^{2K_1})' \frac{1}{K'_1 + r^2 K''_1}). \quad (2.53)$$

where the primes denote ∂_{r^2} . Dividing the 2nd equation above by the 1st,

$$\frac{K''_0}{K'_0} + \frac{K''_1}{K'_1} = -\frac{2}{r^2} \quad (2.54)$$

which means

$$K_1' = \frac{\gamma}{r^4 K_0'}. \quad (2.55)$$

This equation is satisfied for AdS with $\gamma = \frac{1}{4}$. Plugging this back into the 1st equation, we have

$$e^{-2K_1}(K_1' + r^2 K_1'') + \frac{1}{4e\gamma^2} r^8 (K_1')^3 = 0. \quad (2.56)$$

If change of variable, $\frac{1}{2\sqrt{e}\gamma} e^{K_1} = q$, the equation is

$$r^2 q q'' + q q' + r^8 q (q')^3 - r^2 (q')^2 = 0. \quad (2.57)$$

For AdS , we have $q = 2(1 - \frac{1}{r^2})^{\frac{1}{2}}$, where $r^2 > 1$, and $\gamma = \frac{1}{4}$.

Now we look at scaling solutions near a particular r_* region

$$p = q r_*^2, \quad r^2 = r_*^2 (1 + x), \quad (2.58)$$

where $|x| \ll 1$. r_*^2 can still be small compared to the region near the boundary of the droplet space. The equation becomes,

$$p p'' + p p' + p (p')^3 - (p')^2 = 0 \quad (2.59)$$

in which prime means ∂_x , with the inverted equation

$$\ddot{x} - \dot{x}^2 + \frac{1}{p} \dot{x} - 1 = 0 \quad (2.60)$$

where the dot denotes ∂_p .

From the last equation we get a class of solution

$$r^2 = r_*^2 \int_{q_- r_*^2}^{q r_*^2} \frac{c_1 J_1(p) + c_2 Y_1(p)}{c_1 J_0(p) + c_2 Y_0(p)} dp + r_-^2, \quad (2.61)$$

$$r^2 = -r_*^2 \int_{q r_*^2}^{q_+ r_*^2} \frac{c_1 J_1(p) + c_2 Y_1(p)}{c_1 J_0(p) + c_2 Y_0(p)} dp + r_+^2, \quad (2.62)$$

where q_- is the value of q at $r^2 = r_*^2(1 - x_-)$ and q_+ is the value of q at $r^2 = r_*^2(1 + x_+)$, and c_1, c_2 are constants. $J_n(p), Y_n(p)$ denote Bessel function of the first kind and of the second kind, respectively.

So we have

$$K_1' = \frac{1}{r_*^2} \frac{c_1 J_0(p) + c_2 Y_0(p)}{c_1 J_1(p) + c_2 Y_1(p)} \frac{1}{p}, \quad (2.63)$$

$$K_0' = \frac{\gamma}{r_*^2} \frac{c_1 J_1(p) + c_2 Y_1(p)}{c_1 J_0(p) + c_2 Y_0(p)} p, \quad (2.64)$$

where the primes denote ∂_{r^2} . Then,

$$K_0 = \int_{q-r_*^2}^{qr_*^2} \gamma \left(\frac{c_1 J_1(p) + c_2 Y_1(p)}{c_1 J_0(p) + c_2 Y_0(p)} \right)^2 p dp + c_-, \quad (2.65)$$

$$K_0 = - \int_{qr_*^2}^{q+r_*^2} \gamma \left(\frac{c_1 J_1(p) + c_2 Y_1(p)}{c_1 J_0(p) + c_2 Y_0(p)} \right)^2 p dp + c_+, \quad (2.66)$$

$$K_1 = \log(q) + \frac{1}{2} + \log(2\gamma). \quad (2.67)$$

This solution is for a particular region near r_*^2 , i.e. $r_*^2(1 - x_-) \leq r^2 \leq r_*^2(1 + x_+)$, due to the scaling limit taken in (2.58).

Now we look at region $r \in I_-$: In the range $r^2 \in (r_{2k}^2, r_{2k+1}^2)$, equation (2.47) gives

$$K_0 = c_{2k} + \alpha_1 \log r^2 \quad (2.68)$$

where c_{2k}, α_1 are constants in that interval. Then if $\alpha_1 \neq 0$, (2.48) would give $K_1 = \alpha_2 + \alpha_3 \log r^2$, which is quite different from the form of AdS, near droplet boundary. So this suggests that $\alpha_1 = 0$, and

$$K_0 = c_{2k} \quad (2.69)$$

in the region $r^2 \in (r_{2k}^2, r_{2k+1}^2)$. This conclusion applies rigorously in the case of radial symmetry.

2.5 Non-radially symmetric solutions

In this subsection we study solutions for the K that are not radially symmetric.

We look at the Kahler potential of the form

$$K(z_i, \bar{z}_i, y) = \tilde{K}(a^2, y^2) + y^2 f(z_i, \bar{z}_i), \quad (2.70)$$

$$a^2 = a^2(z_i, \bar{z}_i) \quad (2.71)$$

where $\tilde{K}(a^2, y^2)$ is a regular solution to the radially symmetric equation (2.38)

$$\partial_{a^2} \tilde{K} \partial_{a^2} (a^2 \partial_{a^2} \tilde{K}) = (1 - 4y^2 \partial_{y^2}^2 \tilde{K}) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} \tilde{K}}. \quad (2.72)$$

Adding the $y^2 f(z_i, \bar{z}_i)$ in (2.38) will not change the regularity condition of the solution (2.38), since

$$K(z_i, \bar{z}_i, y)|_{y=0} = \tilde{K}(a^2(z_i, \bar{z}_i), y^2)|_{y=0}. \quad (2.73)$$

This means that if $\tilde{K}(a^2, y^2)$ is a solution that satisfies the boundary conditions (2.8),(2.10), then $K(z_i, \bar{z}_i, y)$ also satisfies the boundary conditions (2.8),(2.10). So the remaining work is to get the reduced equations for $a^2(z_i, \bar{z}_i), f(z_i, \bar{z}_i)$, to guarantee that $K(z_i, \bar{z}_i, y)$ is a solution to (2.7).

The role of $a^2(z_i, \bar{z}_i)$ is to describe the shape of the droplets. The boundary between the two different droplets from the radially symmetric solution is described by

$$a^2(z_i, \bar{z}_j) = \text{const.} \quad (2.74)$$

where the constant could in principle take multiple values as discussed in subsection 2.4.

For example, the *AdS* solution is

$$a^2(z_i, \bar{z}_i) = |z_1|^2 + |z_2|^2, \quad f(z_i, \bar{z}_i) = 0. \quad (2.75)$$

So the boundary between two types of droplets is described by (2.74), e.g. for *AdS*, as

$$a^2(z_i, \bar{z}_i) = |z_1|^2 + |z_2|^2 = 1. \quad (2.76)$$

Changing the $a^2(z_i, \bar{z}_i)$ to a more general function thus changes the droplet shape, while preserving regularity condition. The resulting solution $K(z_i, \bar{z}_i, y)$ is no longer radially symmetric.

We then have

$$(1 - 4y^2 \partial_{y^2}^2 K) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} K} = (1 - 4y^2 \partial_{y^2}^2 \tilde{K}) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} \tilde{K}} e^{2f(z_i, \bar{z}_j)}, \quad (2.77)$$

$$\det \partial_i \partial_{\bar{j}} K \simeq a^4 \partial_i \log a^2 \partial_{\bar{j}} \log a^2 \partial_k \partial_{\bar{l}} \log a^2 (\delta^{i\bar{j}} \delta^{k\bar{l}} - \delta^{i\bar{l}} \delta^{k\bar{j}}) \partial_{a^2} \tilde{K} \partial_{a^2} (a^2 \partial_{a^2} \tilde{K}) + \det(\partial_i \partial_{\bar{j}} \log a^2) (a^2 \partial_{a^2} \tilde{K})^2, \quad (2.78)$$

where in the second equation (2.78) we kept the leading order terms pertaining to \tilde{K} in $\det \partial_i \partial_{\bar{j}} K$ above. So this is an approximation when $y^2 f(z_i, \bar{z}_i)$ is much smaller than $\tilde{K}(a^2, y^2)$ in (2.70), and this is always correct in the small y region.

Comparing (2.72) with (2.77), (2.78), we have

$$e^{2f(z_i, \bar{z}_j)} = a^4 \partial_i \log a^2 \partial_{\bar{j}} \log a^2 \partial_k \partial_{\bar{l}} \log a^2 (\delta^{i\bar{j}} \delta^{k\bar{l}} - \delta^{i\bar{l}} \delta^{k\bar{j}}) \quad (2.79)$$

and

$$\det(\partial_i \partial_{\bar{j}} \log a^2) = 0. \quad (2.80)$$

One can check that the *AdS* expression satisfies the above equations. Once we solve a solution for $a^2(z_i, \bar{z}_i)$ from (2.80), then (2.79) already gives the solution for $f(z_i, \bar{z}_j)$.

We can solve the general function $a^2 = a^2(z_i, \bar{z}_j)$ in two steps. We define

$$\log a^2 = \log \tilde{a}^2 - \log(g^2). \quad (2.81)$$

We look at a stronger condition than (2.80)

$$\det(\partial_i \partial_{\bar{j}} \log \tilde{a}^2) = 0, \quad (2.82)$$

$$(\delta^{i\bar{l}} \delta^{k\bar{j}} - \delta^{i\bar{j}} \delta^{k\bar{l}}) \partial_k \partial_{\bar{l}} \log \tilde{a}^2 \partial_i \partial_{\bar{j}} \log(g^2) - \det(\partial_i \partial_{\bar{j}} \log(g^2)) = 0. \quad (2.83)$$

These two equations add up to the equation (2.80).

One way to treat equation (2.83) is to look at special solutions satisfied by

$$\partial_i \partial_{\bar{j}} \log(g^2) = 0. \quad (2.84)$$

One can look for special solutions

$$\log(g^2) = \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)) \quad (2.85)$$

where p is a function holomorphic in z_i, z_j , and \tilde{f}_p is a parameter, e.g.

$$p(z_i, z_j) = \sum_{l_1, l_2 \in Z^+} \epsilon_{l_1, l_2} (z_1^{l_1} z_2^{l_2}), \quad (2.86)$$

$$p(z_i, z_j) = \sum_{n_1, n_2 \in Z^+} \epsilon_{n_1, n_2} (n_1 \log z_1 + n_2 \log z_2), \quad (2.87)$$

etc, and these functions are superposable in (2.85).

We first let

$$\log \tilde{a}^2 = \log(|z_1|^2 + |z_2|^2) \quad (2.88)$$

which is a solution to (2.82), and the equation (2.83) becomes

$$\frac{1}{(|z_1|^2 + |z_2|^2)^2} z_i \bar{z}_j \partial_i \partial_{\bar{j}} \log(g^2) - \det(\partial_i \partial_{\bar{j}} \log(g^2)) = 0. \quad (2.89)$$

There are several ways to treat this exact equation (2.89). One way is to consider $|\partial_i \partial_{\bar{j}} \log(g^2)| \ll |\partial_i \partial_{\bar{j}} \log \tilde{a}^2|$, then the equation (2.89) in the leading order reduces to

$$z_i \bar{z}_j \partial_i \partial_{\bar{j}} \log(g^2) = 0. \quad (2.90)$$

This equation is a Laplace equation in 4d in the variables $\log z_i, \log \bar{z}_j$. Another way to treat equation (2.83) is to look at special solutions satisfied by $\partial_i \partial_{\bar{j}} \log(g^2) = 0$.

It would also be nice to obtain more general solutions for (2.84), apart from (2.85). It is also possible to solve more general cases in (2.83) when $\partial_i \partial_{\bar{j}} \log(g^2) \neq 0$.

Now we look at the solution

$$\log a^2 = \log(|z_1|^2 + |z_2|^2) - \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)). \quad (2.91)$$

We denote $K_{y=0} = K_d$ in this section, where d refers to the droplet space. So we see that at $y = 0$, $Z = \frac{1}{2}$,

$$K_d = \frac{1}{2} a^2 - \frac{1}{2} \log a^2 \quad (2.92)$$

up to an overall constant shift, where we used that $K_{y=0} = \tilde{K}(a^2, y^2)|_{y=0}$, and also we used the *AdS* expression for \tilde{K} .

So we have

$$K_d = \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \log(|z_1|^2 + |z_2|^2) - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)). \quad (2.93)$$

For example,

$$K_d = \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \log(|z_1|^2 + |z_2|^2) - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p\left(\sum_{l_1, l_2 \in Z^+} \epsilon_{l_1, l_2} z_1^{l_1} z_2^{l_2}\right) + c.c.) \quad (2.94)$$

or

$$K_d = \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \log(|z_1|^2 + |z_2|^2) - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p \sum_{n_1, n_2 \in Z^+} \epsilon_{n_1, n_2} (n_1 \log |z_1|^2 + n_2 \log |z_2|^2) \quad (2.95)$$

etc.

In the case of (2.86), it changes the droplet shape to, according to (2.74),

$$|z_1|^2 + |z_2|^2 = 1 + \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)) = 1 + \tilde{f}_p\left(\sum_{l_1, l_2 \in Z^+} \epsilon_{l_1, l_2} z_1^{l_1} z_2^{l_2}\right) + c.c.) \quad (2.96)$$

where the ϵ 's are small, so it may describe adding small ripples to the solution corresponding to \tilde{a}^2 . l_1, l_2 may be considered as the wave-numbers of the ripples. In the case of (2.87), it may describe adding separate $Z = -\frac{1}{2}$ droplets in the 4d base.

To summarize a little, we considered $\log \tilde{a}^2$ as a solution before adding ripples, and $\log a^2$ as the solution adding ripples on $\log \tilde{a}^2$. The droplet boundary is described by (2.74).

We can also consider more generally

$$\log \tilde{a}^2 = \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2) \quad (2.97)$$

since

$$\det(\partial_i \partial_{\bar{j}} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2)) = 0. \quad (2.98)$$

One can check

$$z_i \bar{z}_j \partial_i \partial_{\bar{j}} \log(g^2) = 0 \quad (2.99)$$

has a special exact solution

$$\log(g^2) = \frac{(\bar{c}_1 z_1 + \bar{c}_2 z_2) + c.c.}{|z_1|^2 + |z_2|^2} \quad (2.100)$$

one can approximately superpose them and get the solution which is correct in the leading order approximation in the large z approximation

$$\log \tilde{a}^2 \simeq \frac{1}{n} \sum_k \log(|z_1 - z_{1,k}|^2 + |z_2 - z_{2,k}|^2) \quad (2.101)$$

or

$$\log \tilde{a}^2 \simeq \int_{\mathcal{D}} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2) \frac{u(z'_i, z'_i)}{n} d^2 z'_1 d^2 z'_2 \quad (2.102)$$

where $\int_{\mathcal{D}} \frac{u}{n} d^2 z'_1 d^2 z'_2 = 1$, and $u(z'_i, z'_i)$ is a density function.

Then

$$\sum_k \frac{1}{n} \frac{(z_i - z_{i,k})(\bar{z}_j - \bar{z}_{j,k})}{(|z_1 - z_{1,k}|^2 + |z_2 - z_{2,k}|^2)^2} \partial_i \partial_{\bar{j}} \log(g^2) - \det(\partial_i \partial_{\bar{j}} \log(g^2)) \simeq 0. \quad (2.103)$$

One can look for special solutions to the stronger equation $\partial_i \partial_{\bar{j}} \log(g) = 0$, e.g. (2.85).

We have then

$$\log a^2 \simeq \sum_k \frac{1}{n} \log(|z_1 - z_{1,k}|^2 + |z_2 - z_{2,k}|^2) - \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)) \quad (2.104)$$

or

$$\log a^2 \simeq \int_{\mathcal{D}} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2) \frac{u}{n} d^2 z'_1 d^2 z'_2 - \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)). \quad (2.105)$$

So we see that at $y = 0$, $Z = \frac{1}{2}$, where we have denoted $K_{y=0} = K_d$ in this section,

$$\begin{aligned} K_d &\simeq \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \sum_k \frac{1}{n} \log(|z_1 - z_{1,k}|^2 + |z_2 - z_{2,k}|^2) \\ &\quad - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)), \end{aligned} \quad (2.106)$$

e.g.

$$\begin{aligned} K_d &\simeq \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \sum_k \frac{1}{n} \log(|z_1 - z_{1,k}|^2 + |z_2 - z_{2,k}|^2) \\ &\quad - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p\left(\sum_{l_1, l_2 \in Z^+} \epsilon_{l_1, l_2} (z_1^{l_1} z_2^{l_2}) + c.c.\right) \end{aligned} \quad (2.107)$$

or

$$\begin{aligned} K_d &\simeq \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \sum_k \frac{1}{n} \log(|z_1 - z_{1,k}|^2 + |z_2 - z_{2,k}|^2) \\ &\quad - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p\left(\sum_{n_1, n_2 \in Z^+} \epsilon_{n_1, n_2} (n_1 \log z_1 + n_2 \log z_2) + c.c.\right) \end{aligned} \quad (2.108)$$

etc; or, in the continuous approximation

$$\begin{aligned} K_d &\simeq \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2} \int_{\mathcal{D}} \log(|z_1 - z'_1|^2 + |z_2 - z'_2|^2) \frac{u}{n} d^2 z'_1 d^2 z'_2 \\ &\quad - \frac{1}{2}(|z_1|^2 + |z_2|^2 - 1) \tilde{f}_p(p(z_i, z_j) + \bar{p}(\bar{z}_i, \bar{z}_j)). \end{aligned} \quad (2.109)$$

We argue that $K_d(z_i, \bar{z}_i)$ can be interpreted as the effective Hamiltonian of a test eigenvalue in the dual $\mathcal{N}=4$ SYM. We may also interpret $-\partial_i K_d(z_i, \bar{z}_i)$ as the force experienced by the test eigenvalue along the i -direction. For example, for AdS (where the L is restored)

$$K_d(z_i, \bar{z}_i) = \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2}L^2 \log(|z_1|^2 + |z_2|^2), \quad (2.110)$$

$$-\partial_r K_d(z_i, \bar{z}_i) = -r + \frac{L^2}{r}, \quad r \geq L. \quad (2.111)$$

We see that the forces become zero at special $r = L$.

Now we consider the matrix model methods of [1], [7], [8],

$$\mathcal{H} = \frac{1}{2} \text{tr} (|D_0 \Phi_1|^2 + |\Phi_1|^2 + |D_0 \Phi_2|^2 + |\Phi_2|^2) \quad (2.112)$$

where these two matrices are the zero modes of two complex scalars Z, Y in $\mathcal{N}=4$ SYM. The special case of one matrix has been studied in [2], [3], [4] and both the gravity side and gauge side have been matched [2], [3], [4]. When reducing to the eigenvalue basis, the wavefunction of eigenvalues acquires a factor from the measure when integrating out off-diagonal components. The measure terms were derived in [1] for multiple matrices at strong coupling. We can also include interaction terms for multiple matrices and at large N this is relevant for the non-BPS states. Related issues are also discussed in e.g. [49] - [51].

We can consider the wavefunction norm defining an effective Hamiltonian for the eigenvalues as in [1], i.e.

$$\langle \psi | \psi \rangle \sim e^{-2\mathcal{H}_{eff}} \quad (2.113)$$

where ψ is a wavefunction for multiple eigenvalues. The wavefunction norm also gives the probability density function of the eigenvalue distributions.

The effective Hamiltonian from (2.113) [1] is

$$\mathcal{H}_{eff} = \frac{1}{2} \sum_k (|u_{1,k}|^2 + |u_{2,k}|^2) - \frac{1}{2} \sum_{j < k} \log(|u_{1,k} - u_{1,j}|^2 + |u_{2,k} - u_{2,j}|^2) \quad (2.114)$$

where we used a different convention for \mathcal{H}_{eff} in the exponent in (2.113) from the standard notation in [1], to have the mass terms having canonical $\frac{1}{2} |u|^2$ forms, and there is no essential difference. We denote the eigenvalues of the matrices or fields from the gauge theory as $u_{i,k}, \bar{u}_{i,k}$, where i labels the dimensions and k (or j) labels the individual eigenvalues.

We make an identification

$$z_{i,k} = \frac{L}{\sqrt{N}} u_{i,k} \quad (2.115)$$

$$\frac{L^2}{N} \mathcal{H}_{eff} = \frac{1}{2} \sum_k (|z_{1,k}|^2 + |z_{2,k}|^2) - \frac{1}{2} L^2 \sum_{j < k} \frac{1}{N} \log(|z_{1,k} - z_{1,j}|^2 + |z_{2,k} - z_{2,j}|^2) \quad (2.116)$$

up to a shift of constant c , and $c = -\frac{L^2(N-1)}{4} \log \frac{N}{L^2}$ in this case.

We denote the effective Hamiltonian for a test eigenvalue as $H_{eff}(u_i, \bar{u}_i)$, where u_i, \bar{u}_i denotes the test eigenvalue. So the effective Hamiltonian for a test eigenvalue, $H_{eff}(u_i, \bar{u}_i)$, is given by, where the L is restored,

$$\begin{aligned} \frac{L^2}{N} H_{eff} &= \frac{1}{2} (|z_1|^2 + |z_2|^2) - \frac{1}{2} L^2 \sum_j \frac{1}{N} \log(|z_1 - z_{1,j}|^2 + |z_2 - z_{2,j}|^2) \\ &\simeq \frac{1}{2} (|z_1|^2 + |z_2|^2) - \frac{1}{2} L^2 \log (|z_1|^2 + |z_2|^2). \end{aligned} \quad (2.117)$$

The most probable eigenvalue density distribution has an S^3 symmetry, and the last term is a sum over the eigenvalues, whose leading term is such that the overall force is exerted from the origin. We have used the notation \mathcal{H}_{eff} to denote the effective Hamiltonian of a system of N eigenvalues, while we have used $H_{eff}(u_i, \bar{u}_i)$ to denote the effective Hamiltonian of one test eigenvalue (u_i, \bar{u}_i) .

So we have

$$K_d\left(\frac{L}{\sqrt{N}}u_i, \frac{L}{\sqrt{N}}\bar{u}_i\right) = \frac{L^2}{N} H_{eff}(u_i, \bar{u}_i) = -\frac{1}{2} \frac{L^2}{N} \log \langle \psi_b(u_i, \bar{u}_i) | \psi_b(u_i, \bar{u}_i) \rangle \quad (2.118)$$

where $\frac{L}{\sqrt{N}}u_i = z_i$, and the $\psi_b(u_i, \bar{u}_i)$ denotes the reduced wavefunction for the test eigenvalue, and H_{eff} here is the effective Hamiltonian for the test eigenvalue. We have $i = 1, 2$ for 1/4 BPS case, in this section. However, this interpretation that we have suggested here may not be the only interpretation.

For the excited states, the wavefunction changes, and thus the effective Hamiltonian also changes and can be defined via e.g. (2.113) [1], [7].

Finally, we suggest that the geometries corresponding to $\log a^2$, e.g. in (2.81), (2.86), would correspond to the operators of the schematic form

$$O \sim \prod_{l_1, l_2} e^{\text{tr}(Z^{l_1} Y^{l_2})} O_\chi \quad (2.119)$$

where O_χ , or the superposition thereof, may be the operator dual to the geometry corresponding to $\log \tilde{a}^2$. More generally, O_χ may be related to solutions to (4.4). Similar 1/4 BPS operators of the form related to O_χ have been analyzed in e.g. [49] - [53]. The first factor can be considered as the coherent states, whose expansions are superpositions of polynomials of traces. The ripples can be regarded as the collective phenomenon of many eigenvalues. The ripples here are analogous to the ripples in the 1/2 BPS case, e.g. [1], [14], [16].

3 1/8 BPS geometries with J_1, J_2, J_3

3.1 General ansatz

In this section we study the 1/8 BPS states with $U(1)_t \times SO(4)$ symmetry. The geometries corresponding to such BPS states have been studied in [29] and in [30], [36], [31] (see also

related discussion [34], [33]), which is a $R_t \times S^3$ fibration over 6d Kähler base. As argued in [30], the S^3 can shrink smoothly on the location of 5d surfaces in the 6d base.

In the conventions of [29], and [30], we can write the ten dimensional ansatz as

$$\begin{aligned} ds_{10}^2 &= -e^{2\alpha}(dt + \omega)^2 + e^{-2\alpha}(2\partial_i\partial_{\bar{j}}K)dz^i d\bar{z}^{\bar{j}} + e^{2\alpha}d\Omega_3^2, \\ F_5 &= (d[e^{4\alpha}(dt + \omega)] - 2i\eta\partial\bar{\partial}K) \wedge \Omega_3 + \text{dual}, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} 2\eta d\omega &= \mathcal{R}, \quad e^{-4\alpha} = -\frac{1}{8}R, \\ \square_6 e^{-4\alpha} &= \frac{1}{8}(R_{ab}R^{ab} - \frac{1}{2}R^2). \end{aligned} \quad (3.2)$$

R and \mathcal{R} are the Ricci scalar and Ricci form of the 6d base $h_{ab}dx^a dx^b = 2\partial_i\partial_{\bar{j}}K dz^i d\bar{z}^{\bar{j}}$, where \square_6 is with respect to the metric h_{ab} , and $K = K(z_i, \bar{z}_{\bar{i}})$, $i = 1, 2, 3$, is the Kähler potential of the 6d base. The y direction for the 1/8 BPS ansatz can be considered as $y^2 = e^{2\alpha}$.

One can also consider two types of 1/8 BPS states with the above ansatz. One type is the states with three R-charges J_1, J_2, J_3 , which will be discussed in this section. Another type is the states with R-charge J and AdS spins or $SO(4)$ spins S_1, S_2 , which will be discussed in section 5. The first case corresponds to that the S^3 in the ansatz (3.1) is in the AdS directions, while the second case corresponds to that the S^3 in the ansatz (3.1) is in the S^5 directions.

For the first 1/8 BPS sector, the dual operator we consider is of the schematic form

$$O \sim \prod_{i=1}^m \text{tr}(Z^{n_{1i}} Y^{n_{2i}} X^{n_{3i}}) \quad (3.3)$$

where Z, Y, X are three complex scalars of $\mathcal{N}=4$ SYM. The BPS bound is satisfied as

$$\Delta - J_1 - J_2 - J_3 = 0. \quad (3.4)$$

3.2 AdS

Let's first study the most symmetric case, the $AdS_5 \times S^5$. The Kähler potential for the $AdS_5 \times S^5$ solution is [30] (in the unit that the AdS radius $L=1$)

$$K = \frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2) - \frac{1}{2} \log(|z_1|^2 + |z_2|^2 + |z_3|^2) \quad (3.5)$$

i.e.

$$K = \frac{1}{2}|\vec{r}|^2 - \log|\vec{r} - \vec{0}|, \quad (3.6)$$

$$-\nabla K = \frac{1}{|\vec{r} - \vec{0}|} \vec{e}_r - \vec{r} \quad (3.7)$$

where \vec{r} is the radial vector in 6d. As similar to the discussion at the end of section 2.5, we argue that $-\nabla K$ can be considered as an overall force experienced by the test eigenvalue (which includes the forces exerted from all other eigenvalues). In the case of AdS , the overall force is exerted as if from the origin, e.g. in (3.7), and it can be interpreted that the eigenvalues are uniform in angular directions. This enlarged symmetry is related to the $SO(6)$ symmetry of the $\mathcal{N}=4$ SYM.

One can also restore the AdS radius L ,

$$K = \frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2) - \frac{1}{2}L^2 \log(|z_1|^2 + |z_2|^2 + |z_3|^2), \quad (3.8)$$

$$r^2 = |z_1|^2 + |z_2|^2 + |z_3|^2, \\ -\partial_r K(z_i, \bar{z}_i) = \frac{L^2}{r} - r. \quad (3.9)$$

The force balance condition $-\partial_r K(z_i, \bar{z}_i) = 0$ amounts to

$$r = L. \quad (3.10)$$

Now we consider the matrix model methods of [1], [7], [8],

$$\mathcal{H} = \frac{1}{2} \text{tr} (|D_0 \Phi_1|^2 + |\Phi_1|^2 + |D_0 \Phi_2|^2 + |\Phi_2|^2 + |D_0 \Phi_3|^2 + |\Phi_3|^2) \quad (3.11)$$

where these three matrices are the zero modes of three complex scalars Z, Y, X in $\mathcal{N}=4$ SYM. The wavefunction norm defines the effective Hamiltonian [1]

$$\langle \psi | \psi \rangle \sim e^{-2\mathcal{H}_{eff}}, \quad (3.12)$$

$$\mathcal{H}_{eff} = \frac{1}{2} \sum_k (|u_{1,k}|^2 + |u_{2,k}|^2 + |u_{3,k}|^2) - \frac{1}{2} \sum_{j < k} \log(|u_{1,k} - u_{1,j}|^2 + |u_{2,k} - u_{2,j}|^2 + |u_{3,k} - u_{3,j}|^2), \quad (3.13)$$

at strong coupling and large N , where the eigenvalues of the matrices or fields from the gauge theory are denoted as $u_{i,k}, \bar{u}_{i,k}$. We use i to label dimensions, and j, k to label eigenvalues.

Similar to section 2.5, we make an identification

$$z_{i,k} = \frac{L}{\sqrt{N}} u_{i,k} \quad (3.14)$$

$$\frac{L^2}{N} \mathcal{H}_{eff} = \frac{1}{2} \sum_k (|z_{1,k}|^2 + |z_{2,k}|^2 + |z_{3,k}|^2) - \frac{1}{2} L^2 \sum_{j < k} \frac{1}{N} \log(|z_{1,k} - z_{1,j}|^2 + |z_{2,k} - z_{2,j}|^2 + |z_{3,k} - z_{3,j}|^2) \quad (3.15)$$

up to a shift of constant c and $c = -\frac{L^2(N-1)}{4} \log \frac{N}{L^2}$ in this case.

So the effective Hamiltonian for a test eigenvalue $H_{eff}(u_i, \bar{u}_i)$ is, similar to the discussion in section 2.5,

$$\begin{aligned} \frac{L^2}{N} H_{eff} &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2) - \frac{1}{2} L^2 \sum_j \frac{1}{N} \log(|z_1 - z_{1,j}|^2 + |z_2 - z_{2,j}|^2 + |z_3 - z_{3,j}|^2) \\ &\simeq \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2) - \frac{1}{2} L^2 \log (|z_1|^2 + |z_2|^2 + |z_3|^2). \end{aligned} \quad (3.16)$$

The most probable eigenvalue density distribution would have an S^5 symmetry, so the last term is a sum over the eigenvalues, whose net effect in the leading order would be that the force would be exerted from the origin.

So we see that

$$K_d\left(\frac{L}{\sqrt{N}}u_i, \frac{L}{\sqrt{N}}\bar{u}_i\right) = \frac{L^2}{N} H_{eff}(u_i, \bar{u}_i) = -\frac{1}{2} \frac{L^2}{N} \log \langle \psi_b(u_i, \bar{u}_i) | \psi_b(u_i, \bar{u}_i) \rangle \quad (3.17)$$

where $\frac{L}{\sqrt{N}}u_i = z_i$, $K_d = K$, and the $\psi_b(u_i, \bar{u}_i)$ denotes the reduced wavefunction for the test eigenvalue, and H_{eff} here is the effective Hamiltonian for the test eigenvalue. We have $i = 1, 2, 3$ for 1/8 BPS case. This suggests an interpretation that the K_d is the effective Hamiltonian of a test eigenvalue. However, it may not be the only interpretation.

The radial direction $r = \sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2}$ in the 6d space is already the AdS radial direction $\cosh \rho$. The S^5 directions can already be described by $r = 1$. However, in the 6d space, there is more than the $r = 1$ region. When we put an eigenvalue in the location say $r > 1$, it is a configuration that also describes the radial direction of AdS . So the radial direction of the 6d space already describes the radial direction of the AdS . The 6d space may be considered as $R^6 \setminus B^6$.

One can also add ripples on the S^5 . As shown in [30], the gauged supergravity solution of [27] for the scalar-gauge system in 5D corresponds to ellipsoidal droplets, i.e. ripples on the S^5 with lowest possible wave-number.

1/2 BPS geometries were also embedded in the 1/8 BPS ansatz, as performed in [30], demonstrating topologically more nontrivial 5-surfaces.

3.3 More general cases

The Kähler potential for AdS is in the general form

$$K = \frac{1}{2} f(z_1, \bar{z}_1) e^{\Xi(z_i, \bar{z}_i)} - \frac{1}{2} \Xi(z_i, \bar{z}_i) - \frac{1}{2} \log f(z_1, \bar{z}_1) \quad (3.18)$$

$i = 2, 3$. The K of AdS is isotropic in the 6d space. We now relax the symmetry for K on the 6d base space. We can first consider that the z_1, \bar{z}_1 space and z_i, \bar{z}_i space ($i = 2, 3$ here) are not on the equal footing.

The force balance equations are, e.g.

$$-\partial_1 K = \frac{1}{2} \partial_1 f(e^{\Xi(z_i, \bar{z}_i)} - f^{-1}) = 0, \quad (3.19)$$

$$-\partial_i K = \frac{1}{2} \partial_i \Xi(f e^{\Xi(z_i, \bar{z}_i)} - 1) = 0, \quad (3.20)$$

where $i = 2, 3$. The above can be simultaneously solved by

$$f(z_1, \bar{z}_1) = e^{-\Xi(z_i, \bar{z}_i)}. \quad (3.21)$$

In other words, we may consider (3.21) as the constraint giving the shape of the 5-surface argued in [30].

As a special case, one may consider

$$f(z_1, \bar{z}_1) = |z_1|^2 \quad (3.22)$$

which in general means that the Kahler potential preserves S^1 symmetry in the 6d base, and the surface would correspond to

$$|z_1|^2 = e^{-\Xi(z_i, \bar{z}_i)}. \quad (3.23)$$

We may also consider a more general form

$$K = \frac{1}{2} a^2(z_i, \bar{z}_i) - \frac{1}{2} \log a^2(z_i, \bar{z}_i), \quad i = 1, 2, 3 \quad (3.24)$$

i.e. the K is anisotropic in the 6d space. For example,

$$a^2(z_i, \bar{z}_i) = (|z_1|^2 + |z_2|^2 + |z_3|^2) \exp(\tilde{f} \sum_{l_1, l_2, l_3 \in \mathbb{Z}^+} \epsilon_{l_1, l_2, l_3} (z_1^{l_1} z_2^{l_2} z_3^{l_3} + c.c.)). \quad (3.25)$$

The force balance equations are

$$-\partial_i K = \frac{1}{2} (\partial_i \log a^2)(a^2 - 1) = 0, \quad -\partial_i K = \frac{1}{2} (\partial_i \log a^2)(a^2 - 1) = 0 \quad (3.26)$$

which can be simultaneously solved by

$$a^2(z_i, \bar{z}_i) = 1. \quad (3.27)$$

From this, we see that the configuration (3.25) corresponds to adding ripples

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1 - \tilde{f} \sum_{l_1, l_2, l_3 \in \mathbb{Z}^+} \epsilon_{l_1, l_2, l_3} (z_1^{l_1} z_2^{l_2} z_3^{l_3} + c.c.) \quad (3.28)$$

where the ϵ 's are small, which may correspond to the operators schematically

$$O \sim \prod_{l_1, l_2, l_3} e^{\text{tr}(Z^{l_1} Y^{l_2} X^{l_3})}. \quad (3.29)$$

More generally, if we start with

$$a^2(z_i, \bar{z}_i) = \tilde{a}^2(z_i, \bar{z}_i) \exp(\tilde{f} \sum_{l_1, l_2, l_3 \in Z^+} \epsilon_{l_1, l_2, l_3} (z_1^{l_1} z_2^{l_2} z_3^{l_3} + c.c.)), \quad (3.30)$$

the constraint is then

$$\tilde{a}^2(z_i, \bar{z}_i) = 1 - \tilde{f} \sum_{l_1, l_2, l_3 \in Z^+} \epsilon_{l_1, l_2, l_3} (z_1^{l_1} z_2^{l_2} z_3^{l_3} + c.c.). \quad (3.31)$$

This would correspond to adding small ripples on a more general droplet, and may correspond to schematically

$$O \sim \prod_{l_1, l_2, l_3} e^{\text{tr}(Z^{l_1} Y^{l_2} X^{l_3})} O_\chi \quad (3.32)$$

in which case O_χ , or the superposition thereof, may correspond to the droplet without adding the extra ripples.

4 1/4 BPS geometries with S_1, J

4.1 General ansatz

As we discussed at the beginning of section 2, the ansatz (2.1) also describes another set of 1/4 BPS states, corresponding to having an AdS spin or an $SO(4)$ spin S_1 . This case corresponds to that the S^3 in the ansatz (2.1) is to be in the S^5 directions.

For this type of 1/4 BPS sector, the dual operators we will consider here are of the schematic form

$$O \sim \prod_{i=1}^m \text{tr}(D_{++}^{n_{2i}} Z^{n_{1i}}) \quad (4.1)$$

where Z is a complex scalar of $\mathcal{N}=4$ SYM, and D_{++} is a derivative operator on S^3 . The subscripts $+, \dot{+}$ denotes $SU(2)_L$ and $SU(2)_R$ indices. The operator D_{++} carries $SU(2)_L$ and $SU(2)_R$ spins $\frac{1}{2}, \frac{1}{2}$, and thus carries $S_1 = S_L + S_R = 1$, $S_2 = S_L - S_R = 0$. This is a class of 1/4 BPS operators. The BPS bound is satisfied as

$$\Delta - S_1 - J = 0. \quad (4.2)$$

In this case, the Z field (related to z_1 space) and the derivative D_{++} (related to z_2 space) are not on an equal footing, so one does not expect a totally radial symmetry for the K in the 4d base.

One can look at special cases that the Kähler potential is radially symmetric in the z_1 space and z_2 space separately, e.g. one can consider special forms

$$K = K(|z_1|, |z_2|, y). \quad (4.3)$$

In this case, the equation (2.7) is

$$\partial_{|z_1|^2}(|z_1|^2 \partial_{|z_1|^2} K) \partial_{|z_2|^2}(|z_2|^2 \partial_{|z_2|^2} K) - |z_1|^2 |z_2|^2 (\partial_{|z_1|^2} \partial_{|z_2|^2} K)^2 = (1 - 4y^2 \partial_y^2 K) \frac{y}{8\sqrt{e}} e^{2\partial_y K} \quad (4.4)$$

where we also used $e^D = \frac{1}{4\sqrt{e}}$.

If using the gauged ansatz with $n\eta = 2$ in subsection 2.1, the 1/2 BPS geometries, including $AdS_5 \times S^5$, were already embedded in this sector as in section 6.3 and equation (6.73) of [30], when choosing different Z that exchanges the role of two S^3 s.

4.2 AdS, 1st case

The embedding of these states in the ungauged ansatz with $n\eta = 1$ were analyzed in [33], and this subsection analyzes these cases in more details.

As mentioned before, the embedding of both $AdS_5 \times S^5$ and 1/2 BPS geometries in the ansatz describing states with S_1, J were alternatively obtained in the section 6.3 of [30] via the gauged ansatz with $n\eta = 2$.

We first study the $AdS_5 \times S^5$ case with ungauged ansatz $n\eta = 1$. We begin with rewriting the metric on $AdS_5 \times S^5$,

$$ds^2 = -h^{-2}(dt + \omega)^2 + h^2((Z + \frac{1}{2})^{-1} 2\partial_i \partial_{\bar{j}} K dz^i d\bar{z}^{\bar{j}} + dy^2) + y(e^G d\Omega_3^2 + e^{-G}(d\psi)^2) \quad (4.5)$$

$$\begin{aligned} ds^2 = & -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \left[\cos^2 \alpha d\tilde{\phi}_2^2 + d\alpha^2 \right] + d\theta^2 + \cos^2 \theta d\tilde{\phi}_1^2 \\ & + (\sin^2 \theta d\Omega_3^2 + \sinh^2 \rho \sin^2 \alpha d\psi^2). \end{aligned} \quad (4.6)$$

By comparing the $d\Omega_3^2$ factors and $d\psi^2$ factors, we get

$$y = \sin \theta \sinh \rho \sin \alpha, \quad e^G = \frac{\sin \theta}{\sinh \rho \sin \alpha}, \quad (4.7)$$

$$Z = -\frac{1}{2} \frac{\sinh^2 \rho \sin^2 \alpha - \sin^2 \theta}{(\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta)}, \quad h^{-2} = \sinh^2 \rho \sin^2 \alpha + \sin^2 \theta. \quad (4.8)$$

Similar to the 1/2 BPS case [4], we make a shift of the angular variables $\tilde{\phi}_1 = \phi_1 + t$, $\tilde{\phi}_2 = \phi_2 - t$. This gives the mixing terms between time and angular variables ϕ_1, ϕ_2 on the base where $z_1 = |z_1| e^{i\phi_1}$, $z_2 = |z_2| e^{i\phi_2}$. So the one form ω is

$$\omega = h^2(-\cos^2 \theta d\phi_1 + \sinh^2 \rho \cos^2 \alpha d\phi_2) \quad (4.9)$$

and

$$dt + \omega = dt + h^2(-\cos^2 \theta d\phi_1 + \sinh^2 \rho \cos^2 \alpha d\phi_2). \quad (4.10)$$

The metric in the y direction and 4d base is

$$\begin{aligned}
& h^2((Z + \frac{1}{2})^{-1} 2\partial_i \partial_{\bar{j}} K dz^i d\bar{z}^{\bar{j}} + dy^2) = h^2 dy^2 \\
& + \frac{2}{\sin^2 \theta} (\partial_1 \partial_{\bar{1}} K (d|z_1|^2 + |z_1|^2 d\phi_1^2) + \partial_2 \partial_{\bar{2}} K (d|z_2|^2 + |z_2|^2 d\phi_2^2) + \partial_1 \partial_{\bar{2}} K dz_1 d\bar{z}_2 + \partial_2 \partial_{\bar{1}} K dz_2 d\bar{z}_1)
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
= & d\rho^2 + d\theta^2 + \sinh^2 \rho d\alpha^2 + \frac{\cos^2 \theta (\sinh^2 \rho \sin^2 \alpha + 1)}{(\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta)} d\phi_1^2 + \frac{\sinh^2 \rho \cos^2 \alpha (\sinh^2 \rho + \sin^2 \theta)}{(\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta)} d\phi_2^2 \\
& + \frac{2 \sinh^2 \rho \cos^2 \alpha \cos^2 \theta}{(\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta)} d\phi_1 d\phi_2.
\end{aligned} \tag{4.12}$$

The ρ, θ, α directions are orthogonal to ϕ_1, ϕ_2 . The θ, α angles are orthogonal to each other. We may therefore assume the change of variables from ρ, θ, α to $|z_1|, |z_2|$:

$$|z_1| = |z_1|(\rho, \theta), \quad |z_2| = |z_2|(\rho, \alpha). \tag{4.13}$$

By comparing $g_{\phi_1 \phi_1}$ we get

$$\partial_1 \partial_{\bar{1}} K = \frac{\cos^2 \theta \sin^2 \theta}{2|z_1|^2} \frac{\sinh^2 \rho \sin^2 \alpha + 1}{\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta}. \tag{4.14}$$

While by comparing $g_{\phi_2 \phi_2}$ we have

$$\partial_2 \partial_{\bar{2}} K = \sinh^2 \rho \frac{\cos^2 \alpha (\sinh^2 \rho + \sin^2 \theta)}{(\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta)} \frac{\sin^2 \theta}{2|z_2|^2}. \tag{4.15}$$

By comparing $g_{\theta\theta}$ we get

$$\partial_1 \partial_{\bar{1}} K = \frac{\sin^4 \theta}{2} \frac{\sinh^2 \rho \sin^2 \alpha + 1}{\sinh^2 \rho \sin^2 \alpha + \sin^2 \theta} \left(\frac{\partial |z_1|}{\partial \theta} \right)^{-2}. \tag{4.16}$$

Comparing (4.14), (4.16), we see that $(\log |z_1|)^2 = \log^2(\cosh \rho \cos \theta)$. It implies that we have two possibilities

$$|z_1| = \cosh \rho \cos \theta \tag{4.17}$$

or

$$|z_1| = \frac{1}{\cosh \rho \cos \theta} \tag{4.18}$$

which are related by an inversion $|z_1| \rightarrow \frac{1}{|z_1|}$.

We will only consider the first case (4.17) in this subsection. In this case (4.17), when $\sin \theta = 0$, the S^3 shrinks in $|z_1| = \cosh \rho \geq 1$, which is the outside part of a disk in z_1 space. When $\sinh \rho = 0$, the S^1 shrinks in $|z_1| = \cos \theta \leq 1$, which is the inside part of a disk in z_1 space. The second case (4.18) will be considered in subsection 4.3.

By comparing $g_{\alpha\alpha}$ we have

$$\partial_2 \partial_2 K = \frac{\sin^2 \theta \sinh^2 \rho \sin^2 \alpha (\sinh^2 \rho + \sin^2 \theta)}{2 \sinh^2 \rho \sin^2 \alpha + \sin^2 \theta} \left(\frac{\partial |z_2|}{\partial \alpha} \right)^{-2}. \quad (4.19)$$

Comparing (4.15), (4.19), we see $(\log |z_2|)^2 = \log^2(\tanh \rho \cos \alpha)$. It also implies that we have two possibilities, that are related by an inversion of the radial variable. Without loss of generality, we consider the first case

$$|z_2| = \tanh \rho \cos \alpha. \quad (4.20)$$

Now we look at the AdS variables in terms of variables in the ansatz. From (4.8) we have

$$Z = -\frac{1}{2} \frac{a^2 + y^2 - 1}{\sqrt{(a^2 + y^2 - 1)^2 + 4y^2}}, \quad (4.21)$$

$$a^2 = \cos^2 \theta (1 + \sinh^2 \rho \cos^2 \alpha) = |z_1|^2 (1 - |z_2|^2). \quad (4.22)$$

At $y = 0$, $Z = -\frac{1}{2}$ where $a > 1$, and $Z = \frac{1}{2}$ where $a < 1$. So this situation is opposite to the case (2.42) in section 2. We can obtain ρ, θ, α in terms of z_1, z_2, y as e.g.

$$\sinh^2 \rho \sin^2 \alpha = \frac{1}{2} (\sqrt{(a^2 + y^2 - 1)^2 + 4y^2} + (a^2 + y^2 - 1)), \quad (4.23)$$

$$\sin^2 \theta = \frac{1}{2} (\sqrt{(a^2 + y^2 - 1)^2 + 4y^2} - (a^2 + y^2 - 1)), \quad (4.24)$$

$$\cosh^2 \rho = \frac{1 + \sinh^2 \rho \sin^2 \alpha}{(1 - |z_2|^2)}. \quad (4.25)$$

Since

$$Z = -\frac{1}{2} y \partial_y \left(\frac{1}{y} \partial_y K \right) \quad (4.26)$$

we can integrate Z and write

$$K = K_Z + \tilde{K}_0 + y^2 \tilde{K}_1 \quad (4.27)$$

where K_Z is the result of integration of Z in (4.26)

$$\begin{aligned} K_Z &= \frac{1}{4} y^2 \log y^2 + \frac{1}{4} (-b^2 + (y^2 + 2) \log(a^2 + b^2 + y^2 + 1)) \\ &\quad - \frac{1}{4} y^2 \log[2((a^2 + b^2 - 1)b^2 - (a^2 + y^2 + 1)y^2)], \end{aligned} \quad (4.28)$$

$$b^2 = \sqrt{(a^2 + y^2 - 1)^2 + 4y^2}, \quad a^2 = |z_1|^2 (1 - |z_2|^2), \quad (4.29)$$

and \tilde{K}_0, \tilde{K}_1 are two integration constants that do not depend on y .

By integrating the equations from the metric (4.14), (4.15) we get

$$\partial_{|z_1|} K = \frac{-1 + y^2 + |z_1|^2 (1 - |z_2|^2) - \sqrt{(-1 + y^2 + |z_1|^2 (1 - |z_2|^2))^2 + 4y^2}}{4 |z_1|^2}, \quad (4.30)$$

$$\partial_{|z_2|} K = \frac{1 + y^2 - |z_1|^2 (1 - |z_2|^2) + \sqrt{(-1 + y^2 + |z_1|^2 (1 - |z_2|^2))^2 + 4y^2}}{4(1 - |z_2|^2)}. \quad (4.31)$$

Integrating these equations and comparing them with (4.27),(4.28), we obtain

$$\begin{aligned}\tilde{K}_0 + y^2 \tilde{K}_1 &= \frac{1}{4}(a^2 - 2 \log a^2) + \frac{1}{4}y^2(\log |z_1|^2 + \log 2e - \log(1 - |z_2|^2)), \\ a^2 &= |z_1|^2(1 - |z_2|^2).\end{aligned}\tag{4.32}$$

So we get the final expression of K ,

$$\begin{aligned}K &= \frac{1}{4}(a^2 - 2 \log a^2) + \frac{1}{4}y^2(\log |z_1|^2 + \log 2e - \log(1 - |z_2|^2)) \\ &\quad + \frac{1}{4}y^2 \log y^2 + \frac{1}{4}(-b^2 + (y^2 + 2) \log(a^2 + b^2 + y^2 + 1)) \\ &\quad - \frac{1}{4}y^2 \log[2((a^2 + b^2 - 1)b^2 - (a^2 + y^2 + 1)y^2)],\end{aligned}\tag{4.33}$$

$$b^2 = \sqrt{(a^2 + y^2 - 1)^2 + 4y^2}, \quad a^2 = |z_1|^2(1 - |z_2|^2).\tag{4.34}$$

Now we check the one-form equation

$$\omega = \frac{1}{2y} \partial_y [|z_1| \partial_{|z_1|} K d\phi_1 + |z_2| \partial_{|z_2|} K d\phi_2]\tag{4.35}$$

$$= h^2(-\cos^2 \theta d\phi_1 + \sinh^2 \rho \cos^2 \alpha d\phi_2).\tag{4.36}$$

We have

$$\partial_{y^2}(|z_1| \partial_{|z_1|} K) = \frac{1}{2} \left(1 - \frac{y^2 + |z_1|^2(1 - |z_2|^2) + 1}{\sqrt{(y^2 + |z_1|^2(1 - |z_2|^2) - 1)^2 + 4y^2}}\right).\tag{4.37}$$

This exactly agrees with $-h^2 \cos^2 \theta = -(Z + \frac{1}{2}) \frac{\cos^2 \theta}{\sin^2 \theta}$.

We have

$$\partial_{y^2}(|z_2| \partial_{|z_2|} K) = \frac{|z_2|^2}{2(1 - |z_2|^2)} \left(1 + \frac{y^2 + |z_1|^2(1 - |z_2|^2) + 1}{\sqrt{(y^2 + |z_1|^2(1 - |z_2|^2) - 1)^2 + 4y^2}}\right).\tag{4.38}$$

This also exactly agrees with $h^2 \sinh^2 \rho \cos^2 \alpha$.

We also checked that the final expression (4.33) exactly satisfies (2.7).

For AdS ground state, the droplet boundary is described by

$$a^2 = |z_1|^2(1 - |z_2|^2) = 1, \quad |z_2| \leq 1.\tag{4.39}$$

Since

$$|z_2| \leq 1\tag{4.40}$$

the z_2 space is a disk D^2 . The z_1 space has no restriction and is R^2 . Thereby the 4d droplet space is $D^2 \times R^2$. The 3-surface in $D^2 \times R^2$ described by (4.39) has $S^1_{\phi_2} \times S^1_{\phi_1}$ symmetry, where the $S^1_{\phi_2}$ corresponds to the angle in the S^3 directions of AdS . Thereby for more general solutions relaxing this symmetry, one can reduce the spherical symmetry in the AdS directions.

The $S^1 \times S^1$ symmetry of the the ground state in this case, is smaller than the S^3 symmetry of the ground state in the case in section 2.

4.3 AdS, 2nd case

In this subsection, we describe another embedding of the $AdS_5 \times S^5$, which is related to the one in subsection 4.2 via the inversion of $|z_1|$. The ansatz for the metric components is written in the same way as in (4.5), (4.6), (4.7), (4.8) in subsection 4.2.

In this case, we make the angle shift $\tilde{\phi}_1 = \phi_1 - t$, $\tilde{t} = t + 2\phi_1$, $\tilde{\phi}_2 = \phi_2 - t$, so

$$d\tilde{t} + \tilde{\omega} = dt + \omega, \quad \tilde{\omega} = \omega - 2d\phi_1 \quad (4.41)$$

$$\tilde{\omega} = h^2(\cos^2 \theta d\phi_1 + \sinh^2 \rho \cos^2 \alpha d\phi_2) - 2d\phi_1 \quad (4.42)$$

$$= \frac{1}{2y} \partial_y [|z_1| \partial_{|z_1|} K d\phi_1 + |z_2| \partial_{|z_2|} K d\phi_2]. \quad (4.43)$$

We have

$$d\tilde{t} + \tilde{\omega} = dt + \omega = dt + h^2(\cos^2 \theta d\phi_1 + \sinh^2 \rho \cos^2 \alpha d\phi_2). \quad (4.44)$$

So comparing the above equation (4.44), with (4.10), the net change is the reversal of the sign of $d\phi_1$ as in (4.44). The shift in $\tilde{\omega}$ is only an exact form. The symbolic expressions for (4.14), (4.15), (4.16), (4.19) in terms of the ρ, θ, α variables have no change. The change is that we select the solution (4.18), i.e.

$$|z_1| = \frac{1}{\cosh \rho \cos \theta}. \quad (4.45)$$

In this case (4.45), when $\sin \theta = 0$, the S^3 shrinks in $|z_1| = \frac{1}{\cosh \rho} \leq 1$, which is the inside part of a disk in z_1 space. When $\sinh \rho = 0$, the S^1 shrinks in $|z_1| = \frac{1}{\cos \theta} \geq 1$, which is the outside part of a disk in z_1 space. The inversion $|z_1| \rightarrow \frac{1}{|z_1|}$ exchanges the inside part with the outside part of the disk in z_1 space.

The expression of $a^2 = \cos^2 \theta (1 + \sinh^2 \rho \cos^2 \alpha)$ in terms of ρ, θ, α variables, as in (4.22), has no change, and its dependence on z_1 changes to

$$a^2 = \cos^2 \theta (1 + \sinh^2 \rho \cos^2 \alpha) = \frac{(1 - |z_2|^2)}{|z_1|^2}. \quad (4.46)$$

After using the new expression (4.45) in (4.14), (4.15), and performing integrations of (4.14), (4.15), we get

$$\partial_{|z_1|} K = \frac{-\frac{(1-|z_2|^2)}{|z_1|^2} + 1 - 5y^2 + \sqrt{(y^2 + \frac{(1-|z_2|^2)}{|z_1|^2} - 1)^2 + 4y^2}}{4|z_1|^2}, \quad (4.47)$$

$$\partial_{|z_2|} K = \frac{-\frac{(1-|z_2|^2)}{|z_1|^2} + 1 + y^2 + \sqrt{(y^2 + \frac{(1-|z_2|^2)}{|z_1|^2} - 1)^2 + 4y^2}}{4(1 - |z_2|^2)}. \quad (4.48)$$

$$K_Z = \frac{1}{4}y^2 \log y^2 + \frac{1}{4}(-b^2 + (y^2 + 2) \log(a^2 + b^2 + y^2 + 1)) - \frac{1}{4}y^2 \log[2((a^2 + b^2 - 1)b^2 - (a^2 + y^2 + 1)y^2)], \quad (4.49)$$

$$b^2 = \sqrt{(a^2 + y^2 - 1)^2 + 4y^2}, \quad a^2 = \frac{(1 - |z_2|^2)}{|z_1|^2}. \quad (4.50)$$

After comparing (4.49),(4.47),(4.48) with (4.27), we obtain

$$\tilde{K}_0 + y^2 \tilde{K}_1 = \frac{1}{4}(a^2 - 2 \log a^2) + \frac{1}{4}y^2(-\log |z_1|^2 + \log 2e - \log(1 - |z_2|^2)) - y^2 \log |z_1|^2. \quad (4.51)$$

So the final result for K is

$$K = \frac{1}{4}(a^2 - 2 \log a^2) + \frac{1}{4}y^2(-\log |z_1|^2 + \log 2e - \log(1 - |z_2|^2)) - y^2 \log |z_1|^2 + \frac{1}{4}y^2 \log y^2 + \frac{1}{4}(-b^2 + (y^2 + 2) \log(a^2 + b^2 + y^2 + 1)) - \frac{1}{4}y^2 \log[2((a^2 + b^2 - 1)b^2 - (a^2 + y^2 + 1)y^2)], \quad (4.52)$$

$$b^2 = \sqrt{(a^2 + y^2 - 1)^2 + 4y^2}, \quad a^2 = \frac{(1 - |z_2|^2)}{|z_1|^2}. \quad (4.53)$$

We now check the one-form equation in (4.42),(4.43). We get from the final expression of K ,

$$\partial_{y^2}(|z_1| \partial_{|z_1|} K) = \frac{\frac{(1 - |z_2|^2)}{|z_1|^2} + 1 + y^2 - \sqrt{(y^2 + \frac{(1 - |z_2|^2)}{|z_1|^2} - 1)^2 + 4y^2}}{2\sqrt{(y^2 + \frac{(1 - |z_2|^2)}{|z_1|^2} - 1)^2 + 4y^2}} - 2, \quad (4.54)$$

$$\partial_{y^2}(|z_2| \partial_{|z_2|} K) = \frac{|z_2|^2 (\frac{(1 - |z_2|^2)}{|z_1|^2} + 1 + y^2 + \sqrt{(y^2 + \frac{(1 - |z_2|^2)}{|z_1|^2} - 1)^2 + 4y^2})}{2(1 - |z_2|^2)\sqrt{(y^2 + \frac{(1 - |z_2|^2)}{|z_1|^2} - 1)^2 + 4y^2}}, \quad (4.55)$$

and they agree with the expression of $\tilde{\omega}$ in (4.42),(4.43). We also checked that the final expression (4.52) exactly satisfies (2.7).

For AdS ground state, the droplet boundary is described by

$$a^2 = (1 - |z_2|^2)/|z_1|^2 = 1 \quad \text{i.e. } |z_2|^2 + |z_1|^2 = 1; \quad |z_2| \leq 1. \quad (4.56)$$

Due to

$$|z_2| \leq 1 \quad (4.57)$$

the droplet space is $D^2 \times R^2$. Although the 3-surface in (4.56) appears to be S^3 in $D^2 \times R^2$, it does not have an exact S^3 symmetry, since it only has the symmetry of $S_{\phi_2}^1 \times S_{\phi_1}^1$. It is related to the first case in (4.39) in subsection 4.2 by exchanging the inside part with the outside part of a disk in the R^2 space of z_1 , i.e. $|z_1| \rightarrow \frac{1}{|z_1|}$.

4.4 Inversion and shift

In subsections 4.2,4.3, we see that there are two embeddings that are related by an inversion $|z_1| \rightarrow \frac{1}{|z_1|}$, and also a shift of Kähler potential at the same time. We denote (4.33),(4.52) as K, \tilde{K} respectively, in this subsection. When comparing (4.33),(4.52), we have

$$\tilde{K}(|\tilde{z}_1|, |z_2|, y) = K(|z_1| = \frac{1}{|\tilde{z}_1|}, |z_2|, y) - y^2 \log |\tilde{z}_1|^2. \quad (4.58)$$

It's straightforward to see that this is correct from the equation (4.4). The (4.4) for K is

$$\frac{1}{|z_1|^2 |z_2|^2} (\partial_{\log|z_1|^2}^2 K \partial_{\log|z_2|^2}^2 K - (\partial_{\log|z_1|^2} \partial_{\log|z_2|^2} K)^2) = (1 - 4y^2 \partial_{y^2}^2 K) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} K}. \quad (4.59)$$

Under the transform $\log |\tilde{z}_1| = -\log |z_1|$, and using (4.58) we have

$$\begin{aligned} \frac{1}{|\tilde{z}_1|^2 |z_2|^2} (\partial_{\log|\tilde{z}_1|^2}^2 \tilde{K} \partial_{\log|z_2|^2}^2 \tilde{K} - (\partial_{\log|\tilde{z}_1|^2} \partial_{\log|z_2|^2} \tilde{K})^2) &= (1 - 4y^2 \partial_{y^2}^2 K) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} K} e^{2\partial_{y^2} (-y^2 \log |\tilde{z}_1|^2)} \\ &= (1 - 4y^2 \partial_{y^2}^2 \tilde{K}) \frac{y}{8\sqrt{e}} e^{2\partial_{y^2} \tilde{K}}, \end{aligned} \quad (4.60)$$

hence \tilde{K} also satisfy the equation.

The metric involves rescaling transformation, but it does not change the gravity background essentially. However, the one-form $\tilde{\omega}$ has a subtle shift by an exact form, as discussed in subsection 4.3. This is due to the shift $-y^2 \log |\tilde{z}_1|^2$ in (4.58), and is also the reason for the shift happened in (4.54).

There is also another shift symmetry: One can add to K a term $-c_1 \log |z_1|^2$ or $-c_2 \log |z_2|^2$, where c_1, c_2 are constants, without changing the equation (4.59), or changing the solution.

4.5 Small y

In this subsection, we analyze the small y behavior of general geometries with S_1 and J .

The small y expansion of the AdS expression in subsection 4.2 for $a^2 = |z_1|^2 (1 - |z_2|^2)$ is as follows:

When $S^3 \rightarrow 0$, $Z = -\frac{1}{2}$, i.e. $a^2 > 1$,

$$K = \frac{1}{4} y^2 \log y^2 + K_0 + y^2 K_1 + (y^2)^2 K_2 + o((y^2)^3), \quad (4.61)$$

$$K_0 = \frac{1}{4} (1 + 2 \log 2), \quad K_1 = \frac{1}{2} (\log |z_1|^2 - \log(a^2 - 1)), \quad K_2 = -\frac{1}{4(a^2 - 1)^2}. \quad (4.62)$$

On the other hand, when $S^1 \rightarrow 0$, $Z = \frac{1}{2}$, i.e. $a^2 < 1$,

$$K = -\frac{1}{4}y^2 \log y^2 + K_0 + y^2 K_1 + (y^2)^2 K_2 + o((y^2)^3), \quad (4.63)$$

$$K_0 = \frac{1}{2}(a^2 - \log a^2) + \frac{1}{4}(-1 + 2 \log 2), \quad K_1 = \frac{1}{2}(1 - \log(1 - |z_2|^2) + \log(1 - a^2)), \quad (4.64)$$

$$K_2 = \frac{1}{4(a^2 - 1)^2}, \quad e^{2K_1} = \frac{e(1 - a^2)}{(1 - |z_2|^2)}. \quad (4.65)$$

While, very similarly, the small y expansion of the *AdS* expression in subsection 4.3 for $a^2 = \frac{(1-|z_2|^2)}{|z_1|^2}$ is:

When $S^3 \rightarrow 0$, $Z = -\frac{1}{2}$, i.e. $a^2 > 1$,

$$K = \frac{1}{4}y^2 \log y^2 + K_0 + y^2 K_1 + (y^2)^2 K_2 + o((y^2)^3), \quad (4.66)$$

$$K_0 = \frac{1}{4}(1 + 2 \log 2), \quad K_1 = \frac{1}{2}(-\log |z_1|^2 - \log(a^2 - 1)) - \log |z_1|^2, \quad K_2 = -\frac{1}{4(a^2 - 1)^2}. \quad (4.67)$$

On the other hand, when $S^1 \rightarrow 0$, $Z = \frac{1}{2}$, i.e. $a^2 < 1$,

$$K = -\frac{1}{4}y^2 \log y^2 + K_0 + y^2 K_1 + (y^2)^2 K_2 + o((y^2)^3), \quad (4.68)$$

$$K_0 = \frac{1}{2}(a^2 - \log a^2) + \frac{1}{4}(-1 + 2 \log 2), \quad K_1 = \frac{1}{2}(1 - \log(1 - |z_2|^2) + \log(1 - a^2)) - \log |z_1|^2, \quad (4.69)$$

$$K_2 = \frac{1}{4(a^2 - 1)^2}, \quad e^{2K_1} = \frac{e(1 - a^2)}{|z_1|^4 (1 - |z_2|^2)}. \quad (4.70)$$

These solutions satisfies the small y equations (2.15),(2.12),(2.13),(2.17),(2.18) analyzed in subsection 2.2, and small y regularity conditions (2.8), (2.10).

Now we study more general solutions in the first case of 4.2. We take both $|z_2|^2 \ll 1$, $|z_1|^2 \ll 1$. Thus $a^2 = |z_1|^2 (1 - |z_2|^2) \ll 1$. This is near the origin of the disk in the z_2 space, as well as near the origin of the $Z = \frac{1}{2}$ droplet region. We have

$$K_0 = -\frac{1}{2} \log |z_1|^2 + \frac{1}{2}(|z_1|^2 + |z_2|^2) + \tilde{K}_0, \quad K_1 = \frac{1}{2} + \tilde{K}_1. \quad (4.71)$$

As discussed in subsection 4.4, one can also add $-\frac{1}{2} \log |z_2|^2$ in K , without changing the solution. So one can also equivalently define

$$K_0 = \frac{1}{2} |z_1|^2 - \frac{1}{2} \log |z_1|^2 + \frac{1}{2} |z_2|^2 - \frac{1}{2} \log |z_2|^2 + \tilde{K}_0, \quad K_1 = \frac{1}{2} + \tilde{K}_1. \quad (4.72)$$

When $S^1 \rightarrow 0$, $Z = \frac{1}{2}$, i.e. $a^2 < 1$, the small y equations (2.17),(2.18) become, in the leading order in tilded variables, as

$$\partial_1 \partial_1 \tilde{K}_0 + \partial_2 \partial_2 \tilde{K}_0 = \tilde{K}_1, \quad (4.73)$$

$$\partial_1 \partial_1 \tilde{K}_1 + \partial_2 \partial_2 \tilde{K}_1 = 0. \quad (4.74)$$

So we get in leading order in small a region

$$\tilde{K}_1 = -\frac{1}{2} \int_{\mathcal{D}} (|z_1 - z'_1|^2 - |z_2 - z'_2|^2) u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2, \quad (4.75)$$

$$\tilde{K}_0 = -\frac{1}{2} \int_{\mathcal{D}} (|z_1 - z'_1|^2 |z_2 - z'_2|^2 - \frac{1}{2} |z_2 - z'_2|^4) u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 \quad (4.76)$$

where $\int_{\mathcal{D}} u(z'_i, \bar{z}'_i) \frac{1}{n} d^2 z'_1 d^2 z'_2 = 1$.

4.6 Large y

In this subsection, we turn to the study of the large y region of the geometries with spin S_1 and J .

We first study the first case as in subsection 4.2. We focus on large y region and make a change of variable

$$K(z_i, \bar{z}_i, y) = \frac{1}{4} y^2 \log y^2 - \frac{1}{2} \log y^2 - \frac{1}{2} y^2 \log(1 - |z_2|^2) - \frac{1}{2y^2} |z_1|^2 (1 - |z_2|^2) + \frac{1}{2} |z_2|^2 + V(z_i, \bar{z}_i, y). \quad (4.77)$$

After cancelling the leading terms in large y in (2.7) exactly, we get a linear equation

$$\frac{1}{(1 - |z_2|^2)^2} 4\partial_1 \partial_{\bar{1}} V + \frac{1}{y^4} (1 - |z_2|^2)^2 4\partial_2 \partial_{\bar{2}} V + y \partial_y \left(\frac{1}{y} \partial_y V \right) = 0. \quad (4.78)$$

If we further take a small $|z_2| \ll 1$ limit, we get

$$4\partial_1 \partial_{\bar{1}} V + y \partial_y \left(\frac{1}{y} \partial_y V \right) + \frac{1}{y^4} 4\partial_2 \partial_{\bar{2}} V = 0. \quad (4.79)$$

We can also change variables to

$$y^2 \Psi = V, \quad (4.80)$$

$$4\partial_1 \partial_{\bar{1}} \Psi + \frac{1}{y^3} \partial_y (y^3 \partial_y \Psi) + \frac{4}{y^4} \partial_2 \partial_{\bar{2}} \Psi = 0. \quad (4.81)$$

The first two terms are given by a 6d Laplacian; while near fixed y , the equation is similar to an 8d Laplace equation.

The analysis for the second case as in subsection 4.3 is very similar, and after the inversion $|z_1| \rightarrow \frac{1}{|z_1|}$, gives the same equations (4.79), (4.81), since the inversion does not affect the large y or small $|z_2|$ limit.

If we expand (4.81) around a certain y_0 , i.e. $y = y_0 + x$, we have

$$\partial_1 \partial_{\bar{1}} \Psi + \frac{1}{y_0^4} \partial_2 \partial_{\bar{2}} \Psi + \frac{1}{4} \partial_x^2 \Psi = 0. \quad (4.82)$$

$$\Psi = \int_{\mathcal{D}} \frac{u(z'_i, \bar{z}'_i) d^2 z'_1 d^2 z'_2}{(|z_1 - z'_1|^2 + |y - y_0|^2 + y_0^4 |z_2 - z'_2|^2)^{\frac{3}{2}}}. \quad (4.83)$$

This solution tells certain information that one can distribute droplets in the z'_i, \bar{z}'_i space, where $i = 1, 2$.

5 1/8 BPS geometries with S_1, S_2, J

5.1 General ansatz

As we discussed at the beginning of section 3, the ansatz (3.1) also describes another set of 1/8 BPS states, corresponding to having AdS spins or $SO(4)$ spins S_1, S_2 . This case corresponds to that the S^3 in the ansatz (3.1) is to be in the S^5 directions.

For this type of 1/8 BPS sector, since we have two $SO(4)$ spins S_1, S_2 , we can consider a number of fields, such as the ones in table (5.1),

	Δ	(S_L, S_R)	(S_1, S_2)	(H_1, H_2, H_3)
Z	1	(0,0)	(0,0)	(0,1,0)
D_{++}	1	$(\frac{1}{2}, \frac{1}{2})$	(1,0)	(0,0,0)
D_{+-}	1	$(\frac{1}{2}, -\frac{1}{2})$	(0,1)	(0,0,0)
λ_{3+}	$\frac{3}{2}$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	(0,1,-1)
λ_{4+}	$\frac{3}{2}$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	(0,0,1)

(5.1)

as well as the field strength F_{++} . We see that $D_{++}D_{+-}Z$ has the same quantum numbers as $\lambda_{3+}\lambda_{4+}$ and ZF_{++} . One can consider representative operators of the schematic form

$$O \sim \prod_{i=1}^m \text{tr}(D_{++}^{n_{2i}} D_{+-}^{n_{3i}} Z^{n_{1i}}) \quad (5.2)$$

where some of the $D_{++}D_{+-}Z$ may be replaced by $\lambda_{3+}\lambda_{4+}$ or ZF_{++} . The BPS bound is satisfied as

$$\Delta - S_1 - S_2 - J = 0. \quad (5.3)$$

We have two spins in AdS direction. In this case, the Z field (related to the z_1 space) and the derivative D_{++}, D_{+-} (related to the z_2, z_3 space) are not on an equal footing, so one does not expect a totally radial symmetry for the K in the 6d base.

5.2 AdS, 1st case

We first study the AdS case. The embedding of the 1/2 BPS geometries in the 1/8 BPS ansatz were obtained in section 5.4 of [30], with the focus on the case that the S^3 of the 1/8 BPS ansatz is in the AdS direction. The general formulas in [30] are also applicable to the case when the S^3 of the 1/8 BPS ansatz is in the S^5 direction.

The embedding is

$$\log(r^2(z_1, \bar{z}_1, y)) = \int^{y^2} \left(Z(z_1, \bar{z}_1, y') + \frac{1}{2} \right) \frac{d(y'^2)}{y'^2}, \quad (5.4)$$

$$\frac{dr}{r} = -i(V_z dz_1 - V_{\bar{z}} d\bar{z}_1) + \frac{Z + \frac{1}{2}}{y} dy, \quad (5.5)$$

$$K(z_1, \bar{z}_1, y^2) = \frac{1}{2} \int^{y^2} \left(Z(z_1, \bar{z}_1, y') + \frac{1}{2} \right) d(y'^2). \quad (5.6)$$

where $r^2 = |z_2|^2 + |z_3|^2$. The integrals will yield expressions up to a term that only depends on z_1 directions, and is independent of y .

Now we embed AdS , in the way that the S^3 in the ansatz is in the S^5 . We consider the Z as,

$$Z = -\frac{1}{2} \frac{|z_1|^2 + y^2 - 1}{\sqrt{(|z_1|^2 + y^2 - 1)^2 + 4y^2}}. \quad (5.7)$$

Comparing to (5.104) of [30], the role of the two S^3 's are switched for this alternative case. The variables are in the unit that the AdS radius $L=1$, and in the above the y is the y variable in the ansatz of the 1/2 BPS geometries.

After performing the y integral, we have

$$K = \frac{1}{4}(y^2 - \sqrt{(|z_1|^2 + y^2 - 1)^2 + 4y^2}) + \frac{1}{2} \log(1 + |z_1|^2 + y^2 + \sqrt{(|z_1|^2 + y^2 - 1)^2 + 4y^2}) + f_1(|z_1|), \quad (5.8)$$

$$r^4 = \left(\frac{2}{1 - |z_1|^2} + \frac{4(|z_1|^2 - y^2 - 1)}{(1 - |z_1|^2)^2(1 + |z_1|^2 + y^2 + \sqrt{(|z_1|^2 + y^2 - 1)^2 + 4y^2})} \right) f_2(|z_1|) \quad (5.9)$$

and we have

$$f_1(|z_1|) = \frac{1}{4}(1 + |z_1|^2) - \frac{1}{2} \log |z_1|^2 - \frac{1}{2} \log 2, \quad (5.10)$$

$$\log f_2(|z_1|) = -\frac{1}{4} \log \left(\frac{2|z_1|^2}{(1 - |z_1|^2)^2} \right). \quad (5.11)$$

Then we get

$$|z_2|^2 = \tanh^2 \rho \cos^2 \alpha, \quad (5.12)$$

$$|z_3|^2 = \tanh^2 \rho \sin^2 \alpha, \quad (5.13)$$

$$|z_1|^2 = \cosh^2 \rho \cos^2 \theta. \quad (5.14)$$

where the ρ, θ, α variables are defined in the same way as in (4.6), and

$$y_{1/8}^2 = 1 - (1 - |z_2|^2 - |z_3|^2)|z_1|^2. \quad (5.15)$$

where $y_{1/8}^2$ is the y^2 variable in the 1/8 BPS ansatz.

And the y^2 variable in the 1/2 BPS ansatz is

$$y_{1/2}^2 = \left(\frac{1}{1 - |z_2|^2 - |z_3|^2} - |z_1|^2 \right) (|z_2|^2 + |z_3|^2). \quad (5.16)$$

Using these variables we simplify K ,

$$K = \frac{1}{2} (1 - |z_2|^2 - |z_3|^2) |z_1|^2 - \frac{1}{2} \log((1 - |z_2|^2 - |z_3|^2) |z_1|^2) \quad (5.17)$$

in the unit with AdS radius $L=1$. The K has an $S^3 \times S^1$ symmetry.

In this case, we have

$$|z_2|^2 + |z_3|^2 \leq 1 \quad (5.18)$$

Hence the z_2, z_3 space is a B^4 , whereas the z_1 space is R^2 . The droplet space thus would be $B^4 \times R^2$.

The surface that $y_{1/8}^2 = 0$ is

$$|z_2|^2 + |z_3|^2 + |z_1|^{-2} = 1 \quad (5.19)$$

or

$$|z_1|^2 - |z_1|^2 |z_2|^2 - |z_1|^2 |z_3|^2 = 1 \quad (5.20)$$

If we focus near the origin of the z_2, z_3 space, this will be near the circle $|z_1| \approx 1$ in z_1 space, and we see that $|z_2|^2 + |z_3|^2 \approx \epsilon^2$ there (where ϵ^2 is small), so the S^3 in z_2, z_3 space shrinks at the boundary of D^2 in z_1 space. The surface in $B^4 \times R^2$ has $S^3 \times S^1$ symmetry.

The equation

$$|z_2|^2 + |z_3|^2 = \tanh^2 \rho = \frac{|z_1|^2 - 1}{|z_1|^2} \quad (5.21)$$

suggests that we have an S^3 and its radius is not constant, and changes with $|z_1|$. This means that the size of the S^3 changes with a radial direction. The radius of S^3 and the radius of S^1 , i.e. $|z_1|$ combine into the overall radial coordinate of the 6d space. In other words, the radius of the S^3 can be viewed as the radial coordinate of the 6d space, projected to the (z_2, z_3) subspace.

If we recover the AdS radius L in the expressions, we have that (5.17) gives

$$K = \frac{1}{2} |z_1|^2 \left(1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2} \right) - \frac{1}{2} L^2 \log \left(1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2} \right) - \frac{1}{2} L^2 \log |z_1|^2. \quad (5.22)$$

Since the S^3 in the z_2, z_3 space is the S^3 of AdS directions, we can have excitations described by general configurations in z_2, z_3 space that correspond to reducing the symmetry of S^3 in AdS directions.

5.3 AdS, 2nd case

In this subsection, we describe another embedding by inversion of $|z_1|$. The initial steps are the same as in (5.4),(5.6),(5.8),(5.9) in subsection 5.2, with the difference of inverting $|z_1|$ as

$$|z_2|^2 = \tanh^2 \rho \cos^2 \alpha, \quad (5.23)$$

$$|z_3|^2 = \tanh^2 \rho \sin^2 \alpha, \quad (5.24)$$

$$|z_1|^2 = \frac{1}{\cosh^2 \rho \cos^2 \theta}. \quad (5.25)$$

$$y_{1/8}^2 = 1 + \frac{(|z_2|^2 + |z_3|^2 - 1)}{|z_1|^2}. \quad (5.26)$$

Then we have in (5.8),(5.9)

$$f_1(|z_1|) = \frac{1}{4} \left(1 + \frac{1}{|z_1|^2}\right) + \frac{1}{2} \log |z_1|^2 - \frac{1}{2} \log 2, \quad (5.27)$$

$$\log f_2(|z_1|) = -\frac{1}{4} \log \left(\frac{2|z_1|^2}{(1 - |z_1|^2)^2} \right). \quad (5.28)$$

Using these variables we simplify K ,

$$K = \frac{1}{2} \frac{(1 - |z_2|^2 - |z_3|^2)}{|z_1|^2} - \frac{1}{2} \log(1 - |z_2|^2 - |z_3|^2) + \frac{1}{2} \log |z_1|^2. \quad (5.29)$$

in the unit with AdS radius $L=1$. It has symmetry $S^3 \times S^1$.

In this case, we also have

$$|z_2|^2 + |z_3|^2 \leq 1. \quad (5.30)$$

Hence the z_2, z_3 space is a B^4 , while the z_1 space is R^2 . The droplet space is also $B^4 \times R^2$.

The surface that $y_{1/8}^2 = 0$ is

$$|z_2|^2 + |z_3|^2 + |z_1|^2 = 1. \quad (5.31)$$

If change of variable $(z_2/z_1 \rightarrow z_2, z_3/z_1 \rightarrow z_3)$,

$$|z_1|^2 + |z_1|^2 |z_2|^2 + |z_1|^2 |z_3|^2 = 1. \quad (5.32)$$

Near the circle $|z_1| \approx 1$, we see that $|z_2|^2 + |z_3|^2 \approx \epsilon^2$, so the S^3 in z_2, z_3 space shrinks at the boundary of D^2 in z_1 space. The surface in $B^4 \times R^2$ may be considered as the fibration of S^3 over a D^2 . The K has an $S^3 \times S^1$ symmetry. Although the equation resembles an S^5 , it may not describe a round sphere. Since the z_2, z_3 space are not on an equal footing with the z_1 space, and there is no symmetry between the z_2, z_3 space and the z_1 space, one may introduce rescalings and (5.31) would appear to describe only deformed S^5 in $B^4 \times R^2$.

The equation

$$|z_2|^2 + |z_3|^2 = \tanh^2 \rho = 1 - |z_1|^2 \quad (5.33)$$

suggests that there is an S^3 and its radius is not constant and changes with $|z_1|$. This means that the size of the S^3 changes with a radial direction, and the radius of S^3 and the $|z_1|$, i.e. the radius of S^1 , combine into the overall radial coordinate of the 6d space. In other words, the radius of S^3 may be viewed as the radial coordinate of the 6d space, projected to the (z_2, z_3) subspace.

5.4 1/2 BPS geometries

We can also embed the 1/2 BPS geometries as the excitations, in the similar way as in e.g. subsection 5.2. The y in this subsection denotes the $y_{1/2}$ of the 1/2 BPS ansatz. We have

$$Z(z_1, \bar{z}_1, y) = -\frac{1}{2} + \frac{y^2}{\pi} \int_D \frac{dx'_1 dx'_2}{[|z_1 - z'_1|^2 + y^2]^2}, \quad (5.34)$$

where the integral is over the areas of the $Z = 1/2$ droplets in the z_1 subspace. From (5.4) we have

$$\log(r^2) = -\frac{1}{\pi} \int_D \frac{dx'_1 dx'_2}{|z_1 - z'_1|^2 + y^2}. \quad (5.35)$$

Integrating (5.34) as in (5.6) gives an expression for the Kahler potential

$$K = -\frac{1}{2\pi} \int_D \left(\frac{y^2}{|z_1 - z'_1|^2 + y^2} - \log[|z_1 - z'_1|^2 + y^2] \right) dx'_1 dx'_2 + \frac{1}{2} - \frac{1}{2} \log |z_1|^2. \quad (5.36)$$

and K in this case only depends on z_1, \bar{z}_1 and r .

In the large $y_{1/2}$ region, the expression of $y_{1/2}$ approaches

$$y_{1/2}^2 = \left(\frac{1}{1 - |z_2|^2 - |z_3|^2} - |z_1|^2 \right) (|z_2|^2 + |z_3|^2), \quad (5.37)$$

which implies that large $y_{1/2}$ corresponds to $|z_2|^2 + |z_3|^2 \rightarrow 1$. At large $y_{1/2}$, we also have

$$\frac{1}{2} \log(|z_1|^2 + y_{1/2}^2) \simeq -\frac{1}{2} \log(1 - |z_2|^2 - |z_3|^2), \quad (5.38)$$

$$-\frac{1}{2} \frac{y_{1/2}^2}{|z_1|^2 + y_{1/2}^2} \simeq -\frac{1}{2} + \frac{1}{2} (1 - |z_2|^2 - |z_3|^2) |z_1|^2. \quad (5.39)$$

5.5 More general cases

Both the above the two cases in subsections 5.2, 5.3 for AdS have the similar form, i.e.

$$K = \frac{1}{2} |z_1|^2 (1 - |z_2|^2 - |z_3|^2) - \frac{1}{2} \log((1 - |z_2|^2 - |z_3|^2) |z_1|^2), \quad (5.40)$$

for the first case and

$$K = \frac{1}{2} |z_1|^{-2} (1 - |z_2|^2 - |z_3|^2) - \frac{1}{2} \log((1 - |z_2|^2 - |z_3|^2) |z_1|^{-2}). \quad (5.41)$$

for another case.

For more general geometries, one change of variable is

$$K = \frac{1}{2}f(z_1, \bar{z}_1)(1 - s(z_i, \bar{z}_i)) - \frac{1}{2}\log(1 - s(z_i, \bar{z}_i)) - \frac{1}{2}\log f(z_1, \bar{z}_1) \quad (5.42)$$

where $i = 2, 3$. This is also similar to (3.18) in subsection 3.3.

One may also introduce the ansatz for adding ripples on the S^3 direction, e.g.

$$s(z_i, \bar{z}_i) = (|z_2|^2 + |z_3|^2)\exp(\tilde{f} \sum_{l_2, l_3 \in \mathbb{Z}^+} \epsilon_{l_2, l_3} (z_2^{l_2} z_3^{l_3} + c.c.)). \quad (5.43)$$

in (5.42).

5.6 Eigenvalue approach

The subsections 4.2, 5.2 suggest that the droplet space of a class of $1/4$ BPS states with S_1, J and of a class of $1/8$ BPS states with S_1, S_2, J may be considered as $D^2 \times R^2$, $B^4 \times R^2$ respectively. In this subsection, we study systems of eigenvalues in the space $B^{2n-2} \times R^2$, $n = 2, 3, 4$, where B^{2n-2} denotes a ball with $2n - 2$ dimensions. The B^{2n-2} may be viewed as R^{2n-2} with a sphere at infinity removed. One can also take a limit focusing on the region near the origin of B^{2n-2} and obtain R^{2n-2} in this limiting procedure.

We first study the case $D^2 \times R^2$. The distance between two points on the disk, in the unit that the radius is 1, is $\rho(u_1, u_2)$, and

$$\sinh^2 \rho(u_1, u_2) = \frac{|u_1 - u_2|^2}{(1 - |u_1|^2)(1 - |u_2|^2)} \quad (5.44)$$

where u is a complex coordinate and $|u|^2 \leq 1$. The potential energy between particles can be written in the form that is the solution to the Poisson equation on the disk,

$$\nabla^2 V = -2\pi\delta(\rho(u_1, u_2)) - 2, \quad (5.45)$$

and one can obtain a solution

$$V = -\frac{1}{2}\log \sinh^2 \rho(u_1, u_2). \quad (5.46)$$

So we see that the force between two particles is given by the potential energy

$$V_{j,k} = -\frac{1}{2}\eta_k \eta_j \log \sinh^2 \rho(u_k, u_j) \quad (5.47)$$

where η_k is the charge of the particle.

For the case of D^2 , with coordinates u_2, \bar{u}_2 , the energy for the system is

$$\mathcal{H}_{eff} = -\frac{1}{2}N \sum_k \log(u_0^2 - |u_{2,k}|^2) - \frac{1}{2} \sum_{j < k} \log(|u_{2,k} - u_{2,j}|^2) \quad (5.48)$$

where k labels individual eigenvalues, and u_0 is a real number, which is the radius of the disk. In the case of B^4 , with two complex coordinates u_2, u_3 , we have a rotational symmetry between u_2, u_3 , so the effective Hamiltonian would be a generalization of (5.48),

$$\mathcal{H}_{eff} = -\frac{1}{2}N \sum_k \log(u_0^2 - |u_{2,k}|^2 - |u_{3,k}|^2) - \frac{1}{2} \sum_{j < k} \log(|u_{2,k} - u_{2,j}|^2 + |u_{3,k} - u_{3,j}|^2) \quad (5.49)$$

where u_0 denotes the radius of B^4 . The last term is due to the similarity with [1].

For the case of $B^4 \times R^2$, due to the symmetry for the measure term, we can have that

$$\begin{aligned} \mathcal{H}_{eff} = & \frac{1}{2} \sum_k |u_{1,k}|^2 - \frac{1}{2}N \sum_k \log\left(1 - \frac{|u_{2,k}|^2}{u_0^2} - \frac{|u_{3,k}|^2}{u_0^2}\right) \\ & - \frac{1}{2} \sum_{j < k} \log(|u_{1,k} - u_{1,j}|^2 + |u_{2,k} - u_{2,j}|^2 + |u_{3,k} - u_{3,j}|^2) \end{aligned} \quad (5.50)$$

up to an overall constant shift, and u_1, \bar{u}_1 denote the R^2 direction.

From the point of view of the wavefunction norm

$$\langle \psi | \psi \rangle \sim e^{-2\mathcal{H}_{eff}} \quad (5.51)$$

the factor $(u_0^2 - |u_{2,k}|^2 - |u_{3,k}|^2)$ will appear in the wavefunction norm, and guarantee that it vanishes at the boundary of B^4 , i.e. $|u_{2,k}|^2 + |u_{3,k}|^2 = u_0^2$. We can also introduce excited state wavefunctions by the multiplication of the ground state wavefunction by additional factors, and change the \mathcal{H}_{eff} . Also, one can obtain an integral expression for the effective Hamiltonian using an eigenvalue density function, replacing the sums. We can also have a limit from (5.50) when setting $u_{3,k} = 0$, which is the $D^2 \times R^2$ case.

Equation (5.50) may also be generalized to higher dimensions for $B^{2n-2} \times R^2$,

$$\mathcal{H}_{eff} = \frac{1}{2} \sum_k |u_{1,k}|^2 - \frac{1}{2}N \sum_k \log\left(1 - \frac{\sum_{i=2}^n |u_{i,k}|^2}{u_0^2}\right) - \frac{1}{2} \sum_{j < k} \log\left(\sum_{i=1}^n |u_{i,k} - u_{i,j}|^2\right) \quad (5.52)$$

where the last term is due to the similarity with [1]. The discussion in this subsection might be applicable for higher n , e.g. B^6 case which may be relevant for 6D theories.

We make a change of variable

$$z_1 = \frac{L}{\sqrt{N}}u_1, \quad z_{2,3} = L\frac{u_{2,3}}{u_0}, \quad K = \frac{L^2}{N}H_{eff}(u_i, \bar{u}_i) \quad (5.53)$$

and (5.50) becomes

$$\begin{aligned} \frac{L^2}{N}\mathcal{H}_{eff} = & \frac{1}{2} \sum_k |z_{1,k}|^2 - \frac{1}{2}L^2 \sum_k \log\left(1 - \frac{|z_{2,k}|^2}{L^2} - \frac{|z_{3,k}|^2}{L^2}\right) \\ & - \frac{1}{2} \frac{L^2}{N} \sum_{j < k} \log(|z_{1,k} - z_{1,j}|^2 + \frac{u_0^2}{N}(|z_{2,k} - z_{2,j}|^2 + |z_{3,k} - z_{3,j}|^2)) \end{aligned} \quad (5.54)$$

up to an overall constant shift $c = -\frac{L^2(N-1)}{4} \log \frac{N}{L^2}$ in this case.

Now we look at the test eigenvalue, and we have the effective Hamiltonian for the test eigenvalue,

$$\begin{aligned} \frac{L^2}{N} H_{eff} &= \frac{1}{2} |z_1|^2 - \frac{1}{2} L^2 \log(1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2}) \\ &\quad - \frac{1}{2} \frac{L^2}{N} \sum_j \log(|z_1 - z_{1,j}|^2 + \frac{u_0^2}{N} (|z_2 - z_{2,j}|^2 + |z_3 - z_{3,j}|^2)) \end{aligned} \quad (5.55)$$

$$\simeq \frac{1}{2} |z_1|^2 - \frac{1}{2} L^2 \log(1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2}) - \frac{1}{2} L^2 \log(|z_1|^2 + \frac{u_0^2}{N} (|z_2|^2 + |z_3|^2)). \quad (5.56)$$

In the last line, we assumed that the eigenvalue distribution has an S^1 symmetry in z_1 space and an S^3 symmetry in z_2, z_3 space, and used the approximation that the overall force for the test eigenvalue would be exerted from the origin, in the leading order, similar to the approximation in (2.117), (3.16).

Now we look at a limiting case

$$|z_2|^2 + |z_3|^2 \ll |z_1|^2 \quad (5.57)$$

and $|z_1|^2$ is finite. We also have $\frac{u_0^2}{N}$ finite. We then have from (5.56),

$$\begin{aligned} \frac{L^2}{N} H_{eff} &\simeq \frac{1}{2} |z_1|^2 (1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2}) - \frac{1}{2} L^2 \log(1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2}) - \frac{1}{2} L^2 \log |z_1|^2 + o(|z_2|^2 + |z_3|^2). \end{aligned} \quad (5.58)$$

The K for AdS in subsection 5.2 is

$$K = \frac{1}{2} |z_1|^2 (1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2}) - \frac{1}{2} L^2 \log(1 - \frac{|z_2|^2}{L^2} - \frac{|z_3|^2}{L^2}) - \frac{1}{2} L^2 \log |z_1|^2. \quad (5.59)$$

We see that this and $\frac{L^2}{N} H_{eff}$ in (5.58) approximately matches in the regime (5.57).

Now we consider (5.50) in the region near the origin of B^4 , i.e. when $\frac{|u_{2,k}|^2}{u_0^2} + \frac{|u_{3,k}|^2}{u_0^2} \ll 1$,

$$\begin{aligned} \mathcal{H}_{eff} &\simeq \frac{1}{2} \sum_k |u_{1,k}|^2 + \frac{1}{2} \sum_k \frac{N}{u_0^2} (|u_{2,k}|^2 + |u_{3,k}|^2) \\ &\quad - \frac{1}{2} \sum_{j < k} \log(|u_{1,k} - u_{1,j}|^2 + |u_{2,k} - u_{2,j}|^2 + |u_{3,k} - u_{3,j}|^2). \end{aligned} \quad (5.60)$$

In this case, we have $B^4 \rightarrow R^4$. We have the mass terms for u_2, u_3 , as given from $\frac{N}{u_0^2}$. If we think of the u_1 as from the scalar Z in $\mathcal{N}=4$ SYM, we argue that these other two eigenvalues u_2, u_3 may come from other fields, or a few higher modes of Z or other fields. The precise realization would depend on which sector of the states that we are focusing on.

5.7 Matrix model method

In this subsection, we consider relating the states with S_1, S_2, J described in subsection 5.1, to the eigenvalues from the $\mathcal{N}=4$ SYM.

We consider writing the scalar Z as

$$Z = \Phi_1 + Y_{1, \frac{1}{2}, \frac{1}{2}}(\Omega)\Phi_2 + Y_{1, \frac{1}{2}, -\frac{1}{2}}(\Omega)\Phi_3 \quad (5.61)$$

where $Y_{l, s_{L3}, s_{R3}}(\Omega)$ denote orthonormalized scalar spherical harmonics on S^3 , in $(\frac{l}{2}, \frac{l}{2})$ representation of $SU(2)_L \times SU(2)_R$, and s_{L3}, s_{R3} are the j_3 of each $SU(2)$. The Ω denotes angular coordinates on the sphere, and the Φ_1, Φ_2, Φ_3 are complex matrices.

The action involving the sector we are considering is

$$S \simeq \int dt \frac{1}{2} \text{tr} (|D_0 \Phi_1|^2 - |\Phi_1|^2 + |D_0 \Phi_2|^2 + |D_0 \Phi_3|^2 - 4|\Phi_2|^2 - 4|\Phi_3|^2) \quad (5.62)$$

where we have integrated out angular coordinates on S^3 , and then rescaled the fields such that they have canonical kinetic terms as in the last line.

We can consider the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \text{tr} (|D_0 \Phi_1|^2 + |\Phi_1|^2 + |D_0 \Phi_2|^2 + |D_0 \Phi_3|^2 + 4|\Phi_2|^2 + 4|\Phi_3|^2). \quad (5.63)$$

One can add more fields, corresponding to larger sectors in the Hilbert space. One can also consider interaction terms or including non-BPS states.

We may map the operators of the 4d theory to the wavefunctions in the reduced 1d model, e.g.

$$\prod_{i=1}^m \text{tr}(D_{++}^{n_{2i}} D_{+-}^{n_{3i}} Z^{n_{1i}}) \rightarrow \prod_{i=1}^m \text{tr}(\Phi_2^{n_{2i}} \Phi_3^{n_{3i}} \Phi_1^{n_{1i}}). \quad (5.64)$$

One can also reduce the model (5.63) in the eigenvalue basis, and then study the quantum mechanics of N eigenvalues in 6d. Their wavefunction norm defines the effective Hamiltonian,

$$\langle \psi | \psi \rangle \sim e^{-2\mathcal{H}_{eff}}. \quad (5.65)$$

The ground state wavefunction gives an effective Hamiltonian

$$\mathcal{H}_{eff} = \frac{1}{2} \sum_k (|u_{1,k}|^2 + 4(|u_{2,k}|^2 + |u_{3,k}|^2)) - \frac{1}{2} \sum_{j < k} \log(|u_{1,k} - u_{1,j}|^2 + |u_{2,k} - u_{2,j}|^2 + |u_{3,k} - u_{3,j}|^2) \quad (5.66)$$

where the complex numbers $u_{i,k}$ (k labels the N eigenvalues) denote the eigenvalues of the three matrices Φ_i , and the last term is due to the insertion of a measure factor in the wavefunction, and was analyzed in [1] for multiple matrices at strong coupling. One can also consider other excited state wavefunctions, which give other effective Hamiltonians.

The effective Hamiltonian also gives a most probable distribution of eigenvalues, when the eigenvalues approximately reaches the equilibrium configuration. For example, from (5.66), the eigenvalue density distribution would have an $S^3 \times S^1$ symmetry.

In the continuum limit of the eigenvalue distribution, one can also approximate the sums in (5.66) by integrals weighted by eigenvalue density function $\rho(u_i, \bar{u}_i)$, so

$$\begin{aligned} \mathcal{H}_{eff} = & \frac{1}{2} \int d^6 u \rho(u_i, \bar{u}_i) (|u_1|^2 + 4(|u_2|^2 + |u_3|^2)) \\ & - \frac{1}{4} \int \int d^6 u d^6 u' \rho(u_i, \bar{u}_i) \rho(u'_i, \bar{u}'_i) \log(|u_1 - u'_1|^2 + |u_2 - u'_2|^2 + |u_3 - u'_3|^2) + \sigma \left(\int d^6 u \rho(u_i, \bar{u}_i) - N \right) \end{aligned} \quad (5.67)$$

where we use the notation $d^6 u = d^2 u_1 d^2 u_2 d^2 u_3$, and $d^2 u_i = \frac{1}{2i} du_i d\bar{u}_i = dx_1 dy_1$. The last term is to enforce the total number of eigenvalues when $\delta_\sigma \mathcal{H}_{eff} = 0$.

We can also reduce (5.66) or (5.63) to the case with the first two matrices Φ_1, Φ_2 only, and it would be relevant to the states with S_1, J as studied in section 4.

We see that the effective Hamiltonian of a test eigenvalue can be considered as

$$H_{test} = \delta_\rho \mathcal{H}_{eff} - \sigma, \quad (5.68)$$

where $\delta_\rho \mathcal{H}_{eff}$ is the derivative of the effective Hamiltonian \mathcal{H}_{eff} with respect to the eigenvalue density. So another interpretation of the K_d as in subsections 2.5, 3.2, 5.6 would be $\delta_\rho \mathcal{H}_{eff} - \sigma$. We can also set $\sigma = 0$ after setting the variation $\delta_\sigma \mathcal{H}_{eff} = 0$.

Now we discuss the eigenvalue distribution. We make variation $\delta_\rho \mathcal{H}_{eff}$,

$$\delta_\rho \mathcal{H}_{eff} = \frac{1}{2} (|u_1|^2 + 4(|u_2|^2 + |u_3|^2)) - \frac{1}{2} \int d^6 u' \rho(u'_i, \bar{u}'_i) \log(|u_1 - u'_1|^2 + |u_2 - u'_2|^2 + |u_3 - u'_3|^2) + \sigma = 0. \quad (5.69)$$

The distribution would have an $S^3 \times S^1$ symmetry. So we may assume

$$\rho(u_i, \bar{u}_i) = \rho(\tilde{r}, |u_1|) \quad (5.70)$$

where $\tilde{r}^2 = |u_2|^2 + |u_3|^2$.

We can make variation of (5.69) with respect to u_i, \bar{u}_i , and obtain the equations for equilibrium configurations. For the equilibrium configuration of the density distribution at low energies, we have an S^3 symmetry in the u_2, u_3 space. If the mass terms for u_2, u_3 and for u_1 were equal, we would have a round S^5 [1]. One can consider this equilibrium configuration (5.69) as deforming the round S^5 due to increasing the mass terms for two complex directions u_2, u_3 . The larger mass terms for u_2, u_3 make the distribution on S^5 no longer uniform in all angles, and they squeeze the S^3 directions. We argue that the eigenvalue distribution form deformed S^5 with $S^3 \times S^1$ symmetry, with the S^3 directions squeezed or decreased in size.

The S^3 here is the S^3 in AdS directions, and for the ground state configuration, we have a round S^3 symmetry in the eigenvalue space. For other 1/8 BPS geometries, one can add ripples on this S^3 . We have therefore argued a possibility that the S^3 symmetry may be related to the distribution of the eigenvalues of the Φ_2, Φ_3 .

6 Discussion

In this paper, we studied four sectors of geometries corresponding to various sectors of 1/4 BPS and 1/8 BPS states in $\mathcal{N}=4$ SYM, with angular momenta in S^5 directions as well as spins in AdS_5 directions.

For the states with angular momenta J_1, J_2 in S^5 directions, we first analyzed the small y equations, and then studied the coupled equations for K_0, K_1 in the $Z=1/2$ region in some detail. We also studied changes of variables, e.g. (2.70), (2.29), which change the Monge-Ampere type equation into other equations, and we studied them in some detail. We also studied the features of adding ripples to the droplets in the $y=0$ hyperplane.

For the states with J_1, J_2, J_3 in S^5 directions, we studied similar issues, including, among other things, analyzing their relations to the eigenvalue pictures.

For the states with spin S_1 in AdS_5 and J in S^5 directions respectively, we first studied multiple embeddings of AdS , and then analyzed small y and large y behaviors of more general solutions. Also, we discussed the inversion symmetry occurred e.g. in the embeddings of AdS .

For the states with spins S_1, S_2 in AdS_5 and J in S^5 directions respectively, we studied similar issues, including, among other things, the multiple embeddings and inversion. We also studied the gauge theory side briefly.

It would be nice to understand many aspects in these topics better in the general context of the AdS/CFT correspondence [48].

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References

- [1] D. Berenstein, JHEP 0601, 125 (2006) [arXiv:hep-th/0507203].
- [2] S. Corley, A. Jevicki and S. Ramgoolam, Adv. Theor. Math. Phys. 5, 809 (2002) [arXiv:hep-th/0111222].
- [3] D. Berenstein, JHEP 0407, 018 (2004) [arXiv:hep-th/0403110].
- [4] H. Lin, O. Lunin and J. M. Maldacena, JHEP 0410, 025 (2004) [arXiv:hep-th/0409174].

- [5] H. Y. Chen, D. H. Correa and G. A. Silva, Phys. Rev. D 76, 026003 (2007) [arXiv:hep-th/0703068].
- [6] S. E. Vazquez, Phys. Rev. D 75, 125012 (2007) [arXiv:hep-th/0612014].
- [7] D. H. Correa and M. Wolf, arXiv:1007.5284 [hep-th].
- [8] D. Berenstein and R. Cotta, JHEP 0704, 071 (2007) [arXiv:hep-th/0702090].
- [9] R. d. M. Koch, JHEP 0811, 061 (2008) [arXiv:0806.0685 [hep-th]].
- [10] H. Lin, A. Morisse and J. P. Shock, JHEP 1006, 055 (2010) [arXiv:1003.4190 [hep-th]].
- [11] S. Gukov and E. Witten, arXiv:0804.1561 [hep-th].
- [12] J. Gomis and S. Matsuura, JHEP 0706, 025 (2007) [arXiv:0704.1657 [hep-th]].
- [13] H. Lin and J. M. Maldacena, Phys. Rev. D 74, 084014 (2006) [arXiv:hep-th/0509235].
- [14] L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V. S. Rychkov, JHEP 0508, 025 (2005) [arXiv:hep-th/0505079].
- [15] G. Mandal, JHEP 0508, 052 (2005) [arXiv:hep-th/0502104].
- [16] Y. Takayama and A. Tsuchiya, JHEP 0510, 004 (2005) [arXiv:hep-th/0507070].
- [17] T. Yoneya, JHEP **0512**, 028 (2005) [arXiv:hep-th/0510114].
- [18] M. M. Caldarelli, D. Klemm and P. J. Silva, Class. Quant. Grav. 22, 3461 (2005) [arXiv:hep-th/0411203].
- [19] T. W. Brown, R. de Mello Koch, S. Ramgoolam and N. Toumbas, JHEP 0703, 072 (2007) [arXiv:hep-th/0611290].
- [20] P. Horava and P. G. Shepard, JHEP 0502, 063 (2005) [arXiv:hep-th/0502127].
- [21] S. Mukhi and M. Smedback, JHEP 0508, 005 (2005) [arXiv:hep-th/0506059].
- [22] H. H. Shieh, G. van Anders and M. Van Raamsdonk, JHEP 0709, 059 (2007) [arXiv:0705.4308 [hep-th]].
- [23] G. T. Horowitz and J. Polchinski, arXiv:gr-qc/0602037.
- [24] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, JHEP 0512, 006 (2005) [arXiv:hep-th/0508023].
- [25] N. V. Suryanarayana, JHEP 0601, 082 (2006) [arXiv:hep-th/0411145].

- [26] V. Balasubramanian, B. Czech, K. Larjo, D. Marolf and J. Simon, JHEP 0712, 067 (2007) [arXiv:0705.4431 [hep-th]].
- [27] Z. W. Chong, H. Lu and C. N. Pope, Phys. Lett. B 614, 96 (2005) [arXiv:hep-th/0412221].
- [28] A. Donos, Phys. Rev. D 75, 025010 (2007) [arXiv:hep-th/0606199].
- [29] N. Kim, JHEP 0601, 094 (2006) [arXiv:hep-th/0511029].
- [30] B. Chen, S. Cremonini, A. Donos, F.-L. Lin, H. Lin, J. T. Liu, D. Vaman and W.-Y. Wen, JHEP 0710, 003 (2007) [arXiv:0704.2233 [hep-th]].
- [31] E. Gava, G. Milanesi, K. S. Narain and M. O’Loughlin, JHEP 0705, 030 (2007) [arXiv:hep-th/0611065].
- [32] A. Donos, JHEP 0705, 072 (2007) [arXiv:hep-th/0610259].
- [33] O. Lunin, JHEP 0809, 028 (2008) [arXiv:0802.0735 [hep-th]].
- [34] J. P. Gauntlett and N. Kim, Commun. Math. Phys. 284, 897 (2008) [arXiv:0710.2590 [hep-th]].
- [35] J. T. Liu, D. Vaman and W. Y. Wen, Nucl. Phys. B 739, 285 (2006) [arXiv:hep-th/0412043].
- [36] J. P. Gauntlett, N. Kim and D. Waldram, JHEP **0704**, 005 (2007) [arXiv:hep-th/0612253].
- [37] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, JHEP 0712, 006 (2007) [arXiv:hep-th/0606087].
- [38] G. Mandal and N. V. Suryanarayana, JHEP 0703, 031 (2007) [arXiv:hep-th/0606088].
- [39] S. Yamaguchi, Int. J. Mod. Phys. A 22, 1353 (2007) [arXiv:hep-th/0601089].
- [40] O. Lunin, JHEP 0606, 026 (2006) [arXiv:hep-th/0604133].
- [41] S. Yamaguchi, JHEP 0605, 037 (2006) [arXiv:hep-th/0603208].
- [42] J. Gomis and F. Passerini, JHEP 0608, 074 (2006) [arXiv:hep-th/0604007].
- [43] J. Gomis and C. Romelsberger, JHEP 0608, 050 (2006) [arXiv:hep-th/0604155].
- [44] N. Drukker, S. Giombi, R. Ricci and D. Trancanelli, JHEP 0704, 008 (2007) [arXiv:hep-th/0612168].
- [45] K. Okuyama and G. W. Semenoff, JHEP 0606, 057 (2006) [arXiv:hep-th/0604209].

- [46] S. A. Hartnoll and S. P. Kumar, JHEP 0608, 026 (2006) [arXiv:hep-th/0605027].
- [47] E. D'Hoker, J. Estes and M. Gutperle, JHEP 0706, 063 (2007) [arXiv:0705.1004 [hep-th]].
- [48] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].
- [49] T. W. Brown, P. J. Heslop and S. Ramgoolam, JHEP 0802, 030 (2008) [arXiv:0711.0176 [hep-th]].
- [50] Y. Kimura and S. Ramgoolam, JHEP 0711, 078 (2007) [arXiv:0709.2158 [hep-th]].
- [51] R. Bhattacharyya, S. Collins and R. d. M. Koch, JHEP 0803, 044 (2008) [arXiv:0801.2061 [hep-th]].
- [52] R. d. M. Koch, N. Ives and M. Stephanou, Phys. Rev. D 79, 026004 (2009) [arXiv:0810.4041 [hep-th]].
- [53] R. de Mello Koch, T. K. Dey, N. Ives and M. Stephanou, JHEP 0908, 083 (2009) [arXiv:0905.2273 [hep-th]].